Mechanical Properties of Concrete and Steel

Reinforced Concrete (RC, also called RCC for Reinforced Cement Concrete) is a widely used construction material in many parts the world. Due to the ready availability of its constituent materials, the strength and economy it provides and the flexibility of its forms, RC is often preferred to steel, masonry or timber in building structures. From a structural analysis and design point of view, RC is a complex composite material. It provides a unique coupling of two materials (concrete, steel) with entirely different mechanical properties.

Stress-Strain Curve for Concrete

Concrete is much stronger in compression than in tension (tensile strength is of the order of one-tenth of compressive strength). While its tensile stress-strain relationship is almost linear [Fig. 1.1(i)], the stress (σ) vs. strain (ε) relationship in compression is nonlinear. Fig. 1.1(ii) shows a typical set of such curves.

These curves are different for various ultimate strengths of concrete, particularly the peak stress and ultimate strain. They consist of an initial relatively elastic portion in which stress and strain are closely proportional, then begin to curve to the horizontal, reaching maximum stress, i.e., the compressive strength f_c', at a strain of approximately 0.002, and finally show a descending branch. They also show that concrete of lower strength are more ductile; i.e., fail at higher strains.

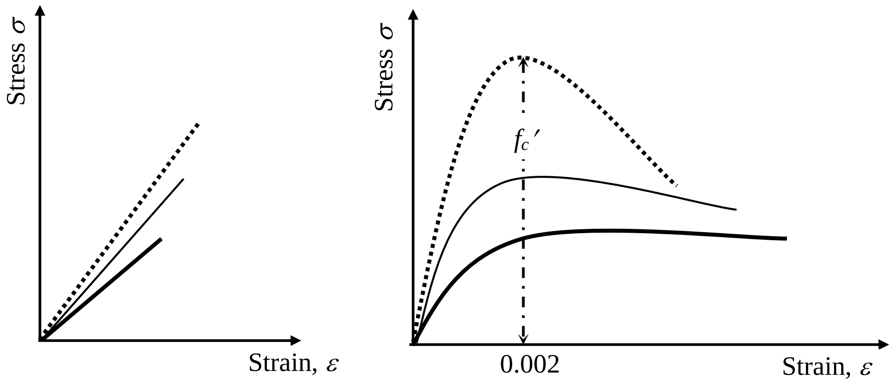


Fig. 1.1: σ-ε curves of Concrete in (i) tension, (ii) compression

The tensile strength as well as the modulus of elasticity of concrete are both proportional to the square-root of its ultimate strength, f_c', and can be approximated by $f_t' = 6\sqrt{f_c'}$………..(1.1), $E_c = 57500\sqrt{f_c'}$………..(1.2) [in psi].

For example, if $f_c' = 3000$ psi, $f_t' \cong 6\sqrt{(3000)} = 330$ psi, and $E_c \cong 57500\sqrt{(3000)} = 3.15 \times 10^6$ psi

Stress-Strain Curve for Steel

Steel is linearly elastic up to a certain stress (called the proportional limit, f_p) after which it reaches yield point (f_y) where the stress remains almost constant despite changes in strain. Beyond f_y, the stress increases again with strain (strain hardening) up to the maximum stress (ultimate strength, f_{ult}) when it decreases until failure at a stress (f_{brk}) quite close to f_y. The typical stress-strain curves for structural steel are shown in Fig. 1.2 (i), which also demonstrate the decreasing ductility of higher-strength steel due to the vanishing yield region. However, the modulus of elasticity (E_s) remains almost constant ($E_s \cong 29000$ ksi) irrespective of strength. The elastic-perfectly-plastic (EPP) model for steel [Fig. 1.2(ii)] assumes the stress to vary linearly with strain up to yield point (f_y) and remain constant beyond that.

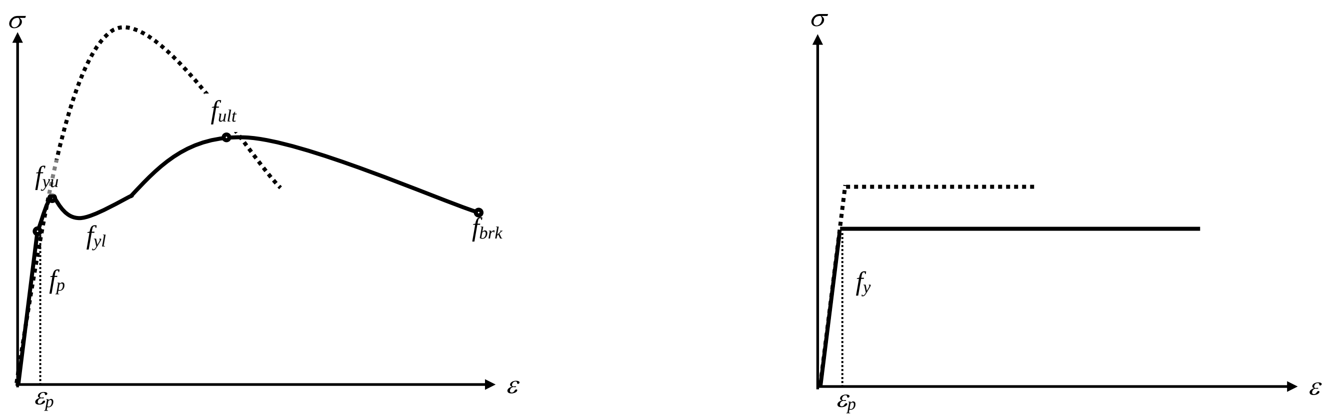


Fig. 1.2: (i) σ-ε curves of Steel (ii) EPP Model

The available bar sizes available in the market are designated by diameters proportional to 1/8-in or millimeters and have the following areas (calculated from $A_s = \pi d^2/4$).

d (No.)	2	3	4	5	6	7	8	9	10
A_s (in^2)	0.05	0.11	0.20	0.31	0.44	0.60	0.79	1.00	1.27
d (mm)	8	10	12	16	19	22	25	28	31
A_s (in^2)	0.08	0.12	0.18	0.31	0.44	0.59	0.76	0.95	1.17

Reinforced Concrete subjected to Axial Force

As Reinforced Concrete (RC) is a complex composite, its structural analysis and design is somewhat different from homogeneous sections. This is valid if it is subjected to axial force (compression/tension) or bending moment.

Axial Compression

When axial (compressive) force is applied, the compressive strain is the same over the entire cross-section and in view of the bonding between concrete and steel, is the same in the two materials. Both concrete and steel behave nearly elastically at low stresses; i.e., which are proportional to strain (corresponding to concrete strain of 0.0003~0.00045 and steel strain of 0.0012~0.0025). Since the stains are equal at a particular point,

$$\varepsilon_c = f_c/E_c \text{ is equal to } \varepsilon_s = f_s/E_s \Rightarrow f_s = (E_s/E_c)\, f_c = n\, f_c \quad \text{....................................(2.1)}$$

where $n = E_s/E_c$, is called the modular ratio.

If A_s = Area of reinforcing bars, A_c = Net area of concrete; i.e., gross area minus A_s, P_c = Axial (compressive) load

$$P_c = f_c A_c + f_s A_s = f_c (A_c + n A_s) = f_c [A_g + (n-1) A_s] \quad \text{....................................(2.2)}$$

To use the basic equations of equilibrium, the structural analysis of RC sections assumes them to be made of a homogeneous material. Instead of changing the modulus of elasticity over the section, the width of various parts is modified proportionately. The stress analysis is made of an *Equivalent* or *Transformed Section*.

Such an analogy cannot be drawn if the concrete exceeds its elastic limit. However, one quantity of particular interest to the structural designer is the ultimate strength, the maximum load which the structure or member will carry. Tests (at different loading conditions and rates) have shown that concrete and steel can be assumed to carry maximum stresses of $0.85f_c'$ and f_y under all circumstances. So the ultimate (nominal) load that the member can safely carry is

$$P_n = 0.85 f_c' A_c + f_y A_s \quad \text{....................................(2.3)}$$

Axial Tension

When the tensile force on a member is small enough for the stress in concrete to be considerably below its tensile strength (f_t'), both concrete and steel behave elastically. In this situation, all the expressions derived for compression are also valid for tension. In particular, the axial force P_c in the earlier equations is now replaced by P_t; i.e.,

$$P_t = f_t [A_g + (n-1) A_s] \quad \text{....................................(2.4)}$$

However, when the load is further increased, concrete reaches its tensile strength and ceases to resist any part of the applied tensile force. So steel is required to resist the entire tensile force. Therefore, at this stage

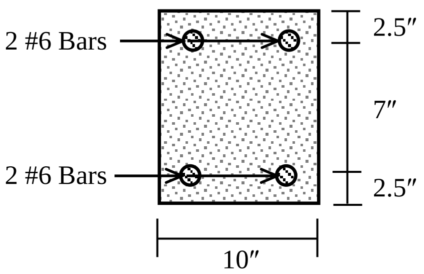
$$P_t = f_s A_s, \text{ and the ultimate tensile force } P_{nt} = f_y A_s \quad \text{....................................(2.5)}$$

Example 2.1

(a) Calculate the ultimate (nominal) compression and tensile force capacity of the RC section shown below.

(b) Also calculate the stresses in concrete and steel when the section is subjected to

(i) one-tenth its ultimate load, (ii) one-half its ultimate load [Given: f_c' = 3 ksi, f_y = 60 ksi].



For f_c' = 3 ksi = 3000 psi, the tensile strength and modulus of elasticity of concrete are approximated by

$f_t' = 6\sqrt{f_c'} = 6\sqrt{(3000)} = 330$ psi

$E_c = 57500\sqrt{f_c'} = 57500\sqrt{(3000)} = 3.15 \times 10^6$ psi

$\therefore$ If $E_s = 29 \times 10^6$ psi, Modular ratio, $n = E_s/E_c \cong 9$

Also $A_g = 10 \times 12 = 120$ in^2, $A_s = 4 \times 0.44 = 1.76$ in^2

$\Rightarrow A_c = 120 - 1.76 = 118.24$ in^2

(a) Eq. (2.3) $\Rightarrow P_n = 0.85f_c'A_c + f_yA_s = 0.85 \times 3 \times 118.24 + 60 \times 1.76 = 301.51 + 105.60 = 407.11$ kips

while Eq. (2.5) $\Rightarrow P_{nt} = f_yA_s = 60 \times 1.76 = 105.60$ kips

(b) (i) If $P_c = P_n/10 = 40.71$ k, $P_c = f_c[A_g + (n-1)A_s] \Rightarrow 40.71 = f_c\,[120 + (9-1)1.76] \Rightarrow f_c = 0.304$ ksi, $f_s = nf_c = 2.73$ ksi

If $P_t = P_{nt}/10 = 10.56$ k, $P_t = f_t[A_g + (n-1)A_s] \Rightarrow 10.56 = f_t\,[120 + (9-1)1.76] \Rightarrow f_t = 0.079$ ksi, $f_s = nf_c = 0.71$ ksi

They correspond to strains of $\varepsilon_c = \varepsilon_s = 2.73/29000 = 9.42 \times 10^{-5}$ and $\varepsilon_s = 0.71/29000 = 2.44 \times 10^{-5}$

(ii) If $P_c = P_n/2 = 203.56$ k $\Rightarrow f_c = 1.52$ ksi, $f_s = 13.66$ ksi $\Rightarrow \varepsilon_c = 13.66/29000 = 4.71 \times 10^{-4}$

If $P_t = P_{nt}/2 = 52.80$ k $\Rightarrow f_t = 0.394$ ksi, $f_s = 3.54$ ksi $\Rightarrow \varepsilon_t = 3.54/29000 = 1.22 \times 10^{-4}$

Both are inappropriate, as concrete will not be elastic up to $\varepsilon_c = 4.71 \times 10^{-4}$, or uncracked up to $\varepsilon_t = 1.22 \times 10^{-4}$

A ‘cracked’ section will give $f_t = 0$ ksi, $f_s = 30$ ksi, while iterations will be needed to obtain appropriate f_c and f_s

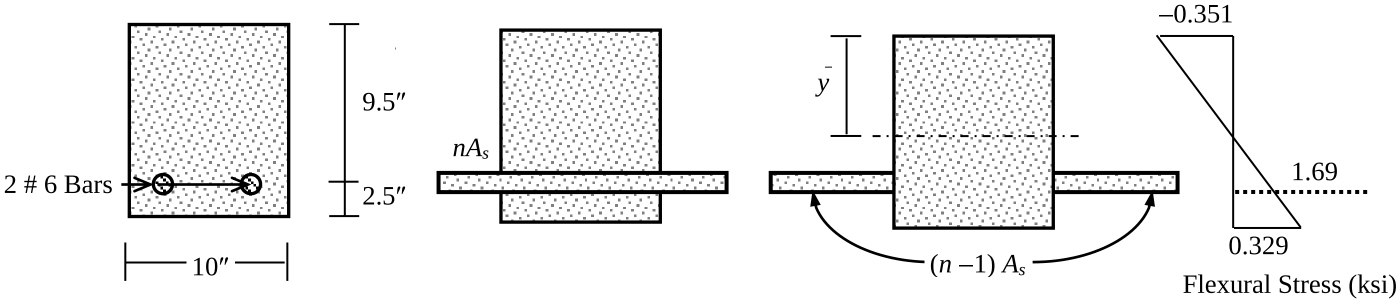
Flexural Stress in Reinforced Concrete

Reinforced Concrete members are designed most often to resist flexural stresses when subjected to bending moments. The following examples illustrate the linearly elastic material behavior of 'uncracked' and 'cracked' RC subjected to flexural stress, when their tensile stresses are below and above the tensile strength of concrete.

<u>'Uncracked' Sections</u>

<u>Example 3.1</u>

Calculate the 'Cracking' positive moment capacity of the RC cross-sectional area shown below.
Also calculate corresponding compressive stress in concrete and tensile stress in steel [Given: $f_c' = 3$ ksi, $f_y = 60$ ksi].



<u>Fig. 3.1: (i) Cross-section, (ii) Equivalent Uncracked Section, (iii) Flexural Stresses</u>

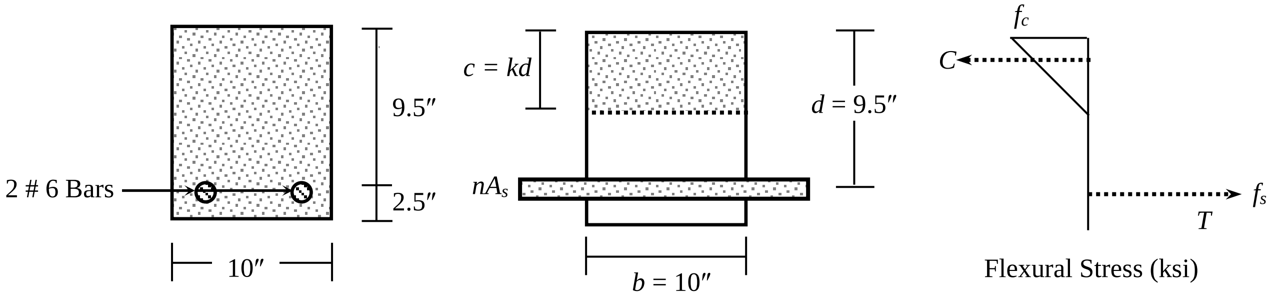
Using $n = 9$, Extra steel area in the Transformed Equivalent section, $(n-1)\,A_s = 8 \times 0.88 = 7.04$ in^2
$\bar{y} = (120 \times 6 + 7.04 \times 9.5)/(120 + 7.04) = 6.19''$
$\therefore$ Moment of Inertia, $\bar{I} = 10 \times 12^3/12 + 120 \times (6 - 6.19)^2 + 7.04 \times (9.5 - 6.19)^2 = 1521.46$ in^4
$\therefore$ Allowable tensile stress in concrete, $f_t = 0.329 = M_{cr} \times (12 - 6.19)/1521.46 \Rightarrow M_{cr} = 86.21$ k-in $= 7.18$ k-ft
Maximum compressive stress in concrete, $f_c = M\,\bar{y}/\,\bar{I} = 86.21 \times 6.19/1521.46 = 0.351$ ksi
Maximum tensile stress in steel, $f_s = 86.21 \times (9.5 - 6.19)/1521.46 \times 9 = 1.69$ ksi

As the 'cracking' moment for the 'unreinforced' section is 78.96 k-in (= 6.58 k-ft), the effect of reinforcing steel is negligible here. The section would crack in tension if subjected to a greater bending moment.

<u>'Cracked' Sections</u>

<u>Example 3.2</u>

Calculate the allowable positive moment in the section shown below, if the allowable compressive stress in concrete is $f_{call} = 1.35$ ksi and allowable tensile stress in steel is $f_{sall} = 24$ ksi.



<u>Fig. 3.2: (i) Cross-section, (ii) Equivalent Uncracked Section, (iii) Flexural Stresses</u>

$\therefore b(kd)^2/2 = nA_s(d-kd) \Rightarrow k^2/2 = n\rho_s\,(1-k) \Rightarrow k = -n\rho_s + \sqrt{[2n\rho_s + (n\rho_s)^2]}$

Tensile force (steel) $T = A_s f_s$, Compressive force (concrete) $C = f_c\, b(kd)/2$
and Moment arm $= d - kd/3 = (1 - k/3)\, d = jd$
$\therefore$ Bending Moment $M_s = T\,jd = A_s f_s\, jd$, and $M_c = C\,jd = f_c\, b(kd)/2\, jd = f_c\,(kj)\, bd^2/2 = R\, bd^2$

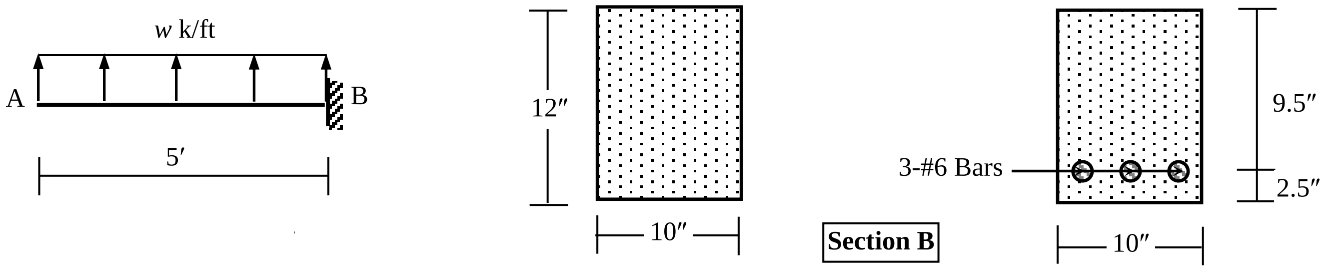
Modular ratio, $n = 9$, Steel Ratio $\rho_s = 0.88/(10 \times 9.5) = 0.0093 \Rightarrow n\rho_s = 0.083$
$\therefore k = -n\rho_s + \sqrt{[2n\rho_s + (n\rho_s)^2]} = 0.333 \Rightarrow c = kd = 3.17''$
Also $j = 1 - k/3 = 0.889 \Rightarrow R = f_{call}\, kj/2 = 1.35 \times 0.333 \times 0.889/2 = 0.20$ ksi
$\therefore M_{call} = R\, bd^2 = 0.20 \times 10 \times 9.5^2 = 180.53$ k-in $= 15.04$ k-ft
and $M_{sall} = A_{sall} f_s\, jd = 0.88 \times 24 \times 0.889 \times 9.5 = 178.34$ k-in $= 14.86$ k-ft
$\therefore$ Allowable bending moment = 14.86 k-ft, corresponding to $f_s = 24$ ksi, and $f_c = (178.34/180.53)\, 1.35 = 1.33$ ksi

Alternatively, Moment of Inertia, $\bar{I} = 10 \times 3.17^3/3 + 9 \times 0.88 \times (9.5 - 3.17)^2 = 423.53$ in^4
$\therefore$ Allowable compressive stress in concrete, $f_{call} = M_{call}\, c/\,\bar{I} \Rightarrow 1.35 = (M_c \times 3.17)/423.53 \Rightarrow M_{call} = 180.53$ k-in
Allowable tensile stress in steel, $f_{sall} = 24$ ksi $= M_{sall} \times (9.5 - 3.17)/423.53 \times 9 \Rightarrow M_{sall} = 178.34$ k-in

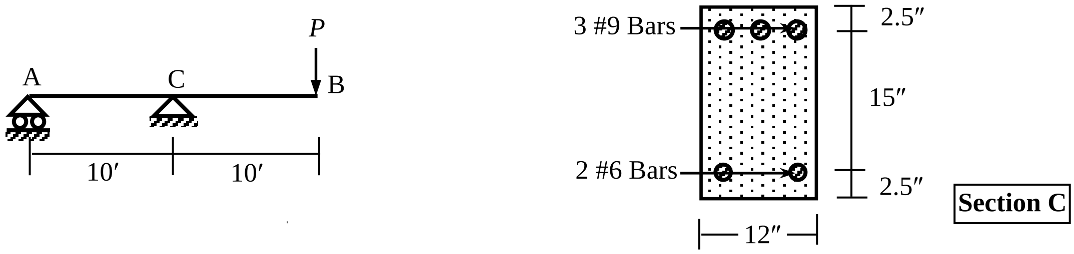
Questions and Problems (1)

1. (i) What is RC? Explain why steel and concrete are used in conjunction in RC.
   (ii) Explain the dependence of stress-strain behavior of concrete and steel on their ultimate strength.
   (iii) What is a 'transformed' RC section? Explain with reference to cracked and uncracked section.
   (iv) Explain the difference between analysis and design of an RC section.
   (v) What is a doubly reinforced RC section? Explain how it differs from a singly reinforced section.

2. In the figures shown below, calculate the maximum distributed load *w* k/ft on the RC cantilever beam AB if the section at B is to remain uncracked. Also calculate the corresponding compressive stress in concrete and tensile stress in steel and check whether they are within the allowable limits
   [Given: Modular ratio = 10, Allowable tensile stress in concrete = 300 psi, Allowable compressive stress in concrete = 1.2 ksi, Allowable tensile stress in steel = 18 ksi].

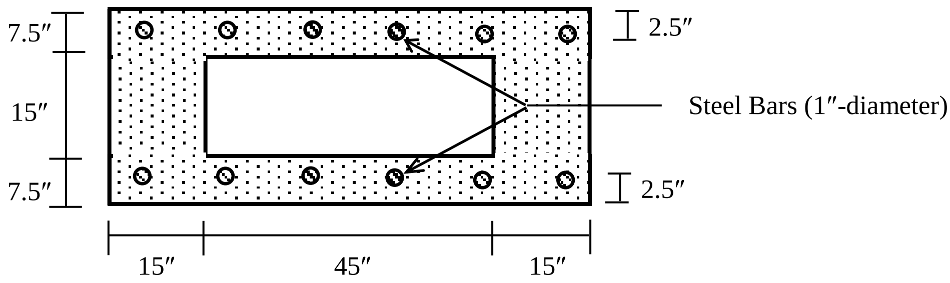


3. Calculate the maximum allowable value of the force *P* if Section C of the RC beam AB shown in the figures below is to remain uncracked. Include the self-weight of the section also while calculating the bending moment.
   [Given: n = 10, Allowable tensile stress in concrete (f_t) = 300 psi, Allowable compressive stress in concrete (f_c) = 1.35 ksi, Allowable tensile stress in steel (f_s) = 18 ksi, Allowable compressive stress in steel (f_s') = 18 ksi].

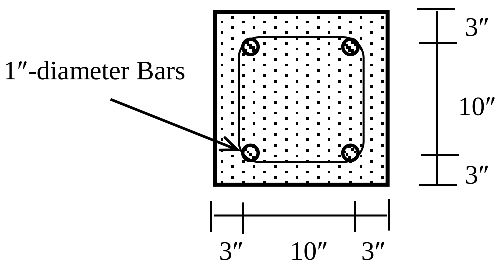


4. Answer Question 3 assuming Section C to have cracked but f_c, f_s and f_s' are to remain within the allowable limits.

5. Calculate the allowable bending moment for the RC section shown below assuming it to be
   (i) uncracked, (ii) cracked due to concrete tension [Given: E_{steel} = 30000 ksi, $E_{concrete}$ = 3000 ksi, allowable f_s = 24 ksi, allowable f_c = 2.0 ksi, allowable f_t = 0.3 ksi].



6. In Question 3, calculate the ultimate value of *P* in beam AB if columns A and C are both made of (16″×16″) RC sections, as shown in the figure below. Also calculate the stresses in concrete and steel when the section is subjected to one-half its ultimate load [Given: f_c' = 4 ksi, f_y = 50 ksi].



Need concrete stress-strain relationship

Flexural Design of Linearly Elastic RC Sections

The discussions so far have dealt with the analysis or review of given RC sections (i.e., given dimensions and reinforcements), and the main objective of such analyses was to calculate their moment or load carrying capacity. In contrast, structural design involves determining the required cross-sectional dimensions and reinforcement areas of such sections so that they can withstand given loads or moments without exceeding given allowable stresses (the main consideration here) or serviceability conditions (e.g., allowable deflection, rotation, curvature).

Fig. 4.1 shows the variation of strain and stress of a rectangular RC section subjected to increasing positive bending moments, demonstrating the (i) uncracked elastic section (upto the tensile cracking of concrete), (ii) cracked elastic section (beyond tensile cracking of concrete, but within the elastic limit of both concrete and steel), (iii) cracked inelastic section (beyond the elastic limit of concrete, upto its crushing strength and yield stress of steel).

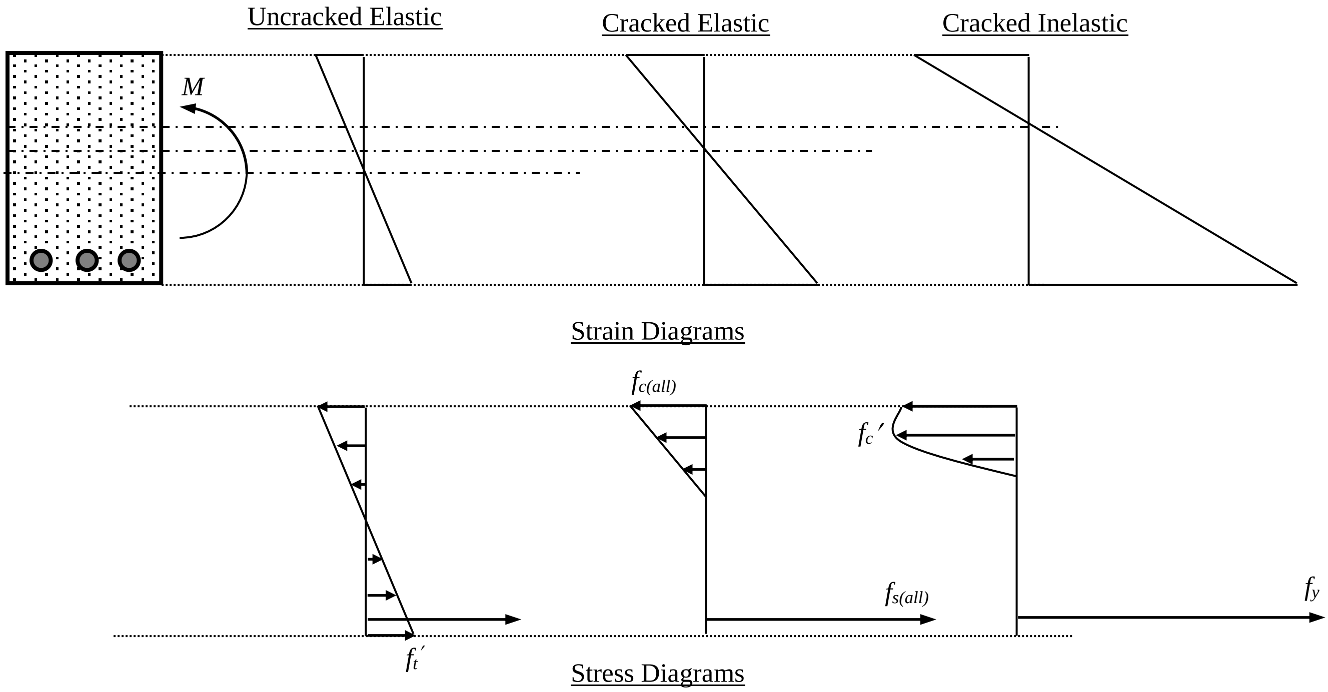
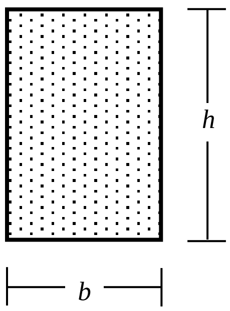


Fig. 4.1: Strain and Stress Diagrams of RC sections (i) Uncracked elastic, (ii) Cracked elastic, (iii) Cracked inelastic

Design of Unreinforced Concrete Section



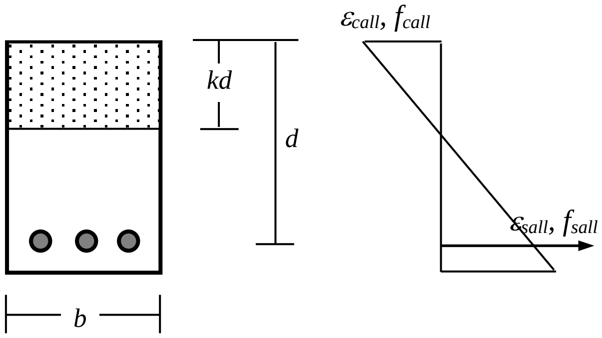
Allowable tensile stress of concrete,

$f_t' = M c / \bar{I} = M (h/2)/(bh^3/12) = 6 M/(bh^2)$

$\Rightarrow h_{req} = \sqrt{(6M/bf_t')}$ ……………………..……..(4.1)

This is the required depth of unreinforced concrete section

Design of Cracked Elastic Section (Working Stress Design)



Strain compatibility $\Rightarrow \varepsilon_c/(kd) = \varepsilon_s/(d - kd)$

$\Rightarrow (f_c/E_c)/k = (f_s/E_s)/(1-k) \Rightarrow n/k = r/(1-k)$

$\Rightarrow k = n/(n + r)$ ……………………... (4.2)

where n = Modular Ratio, $r = f_s/f_c$ = Allowable Stress Ratio

As shown before, Tensile force (steel) $T = A_s f_s$

Compressive force (concrete) $C = f_c\, b(kd)/2$

and Moment arm $= d - kd/3 = jd \Rightarrow j = 1 - k/3$

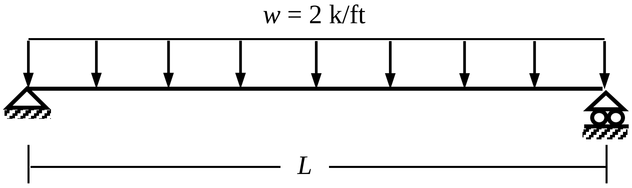
$\therefore$ Bending Moment $M = [f_c(kj)/2]\, bd^2 = R\, bd^2$

$\Rightarrow d_{req} = \sqrt{(M/Rb)}$ ………………………………….……. (4.3)

and $M = T\, jd = A_s f_s\, jd \Rightarrow A_s = M/(jd)$ ……………..….. (4.4)

Example 4.1

Use the WSD Method to design the simply supported singly reinforced RC beams shown below, neglecting their self-weights and assuming



(i) $L = 20'$ [Given: $f_c' = 3$ ksi, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi]

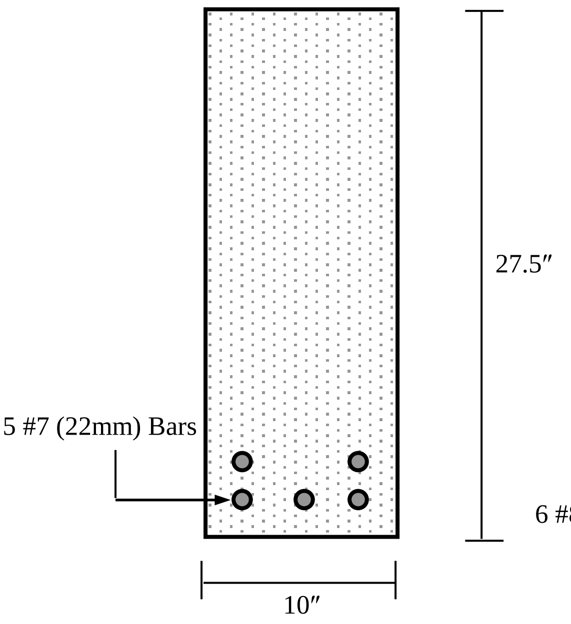
(ii) $L = 30'$ [Given: $f_c' = 4$ ksi, $f_{call} = 1.80$ ksi, $f_{sall} = 24$ ksi]

(i) For $f_c' = 3$ ksi $= 3000$ psi, $n = 9$
$f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi $\Rightarrow r = 20/1.35 = 14.81$
$\therefore k = n/(n + r) = 9/(9 + 14.81) = 0.378 \Rightarrow j = 1 - k/3 = 0.874$
$\therefore R = 0.5\, f_{call}\, kj = 0.5 \times 1.35 \times 0.378 \times 0.874 = 0.223$ ksi

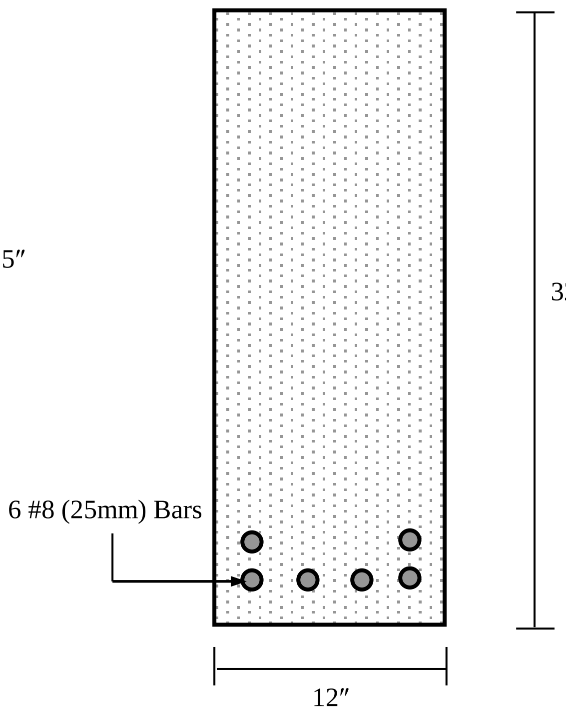
Maximum bending moment $M_{max} = wL^2/8 = 2 \times 20^2/8 = 100$ k-ft
$\therefore$ Assuming beam-width $b = 10''$, $d_{req} = \sqrt{(M_{max}/Rb)} = \sqrt{[100 \times 12/(0.223 \times 10)]} = 23.20'' \Rightarrow h_{req} = 23.20 + 4 = 27.20''$
$\therefore h = 27.5''$, $d = 23.5''$
$\Rightarrow A_s = M_{max}/(f_{sall}\, jd) = 100 \times 12/(20 \times 0.874 \times 23.5) = 2.92$ in^2 $\Rightarrow$ Use 5 #7 or 22-mm bars (in two layers)

(ii) For $f_c' = 4$ ksi $= 4000$ psi, $n = 8$
$f_{call} = 1.8$ ksi, $f_{sall} = 24$ ksi $\Rightarrow r = 24/1.8 = 13.33$
$\therefore k = n/(n + r) = 8/(8 + 13.33) = 0.375 \Rightarrow j = 1 - k/3 = 0.875$
$\therefore R = 0.5\, f_{call}\, kj = 0.5 \times 1.8 \times 0.375 \times 0.875 = 0.295$ ksi

Maximum bending moment $M_{max} = wL^2/8 = 2 \times 30^2/8 = 225$ k-ft
$\therefore$ Assuming beam-width $b = 12''$, $d_{req} = \sqrt{(M_{max}/Rb)} = \sqrt{[225 \times 12/(0.295 \times 12)]} = 27.60'' \Rightarrow h_{req} = 27.60 + 4 = 31.60''$
$\therefore h = 32.0''$, $d = 28.0''$
$\Rightarrow A_s = M_{max}/(f_{sall}\, jd) = 225 \times 12/(24 \times 0.875 \times 28.0) = 4.59$ in^2 $\Rightarrow$ Use 6 #8 or 25-mm bars (in two layers)



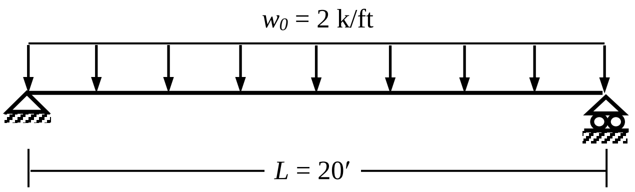
Section (i)

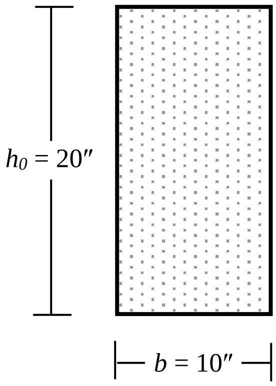


Section (ii)

Example 4.2

Use the WSD Method to design the simply supported singly reinforced RC beam loaded as shown below, in addition to its self-weight [Given: $f_c' = 3$ ksi, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi].





(i) For $f_c' = 3$ ksi = 3000 psi, $n = 9$, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi $\Rightarrow r = 14.81$
$\therefore k = 0.378 \Rightarrow j = 1 - k/3 = 0.874$, $R = 0.5\, f_{call}\, kj = 0.223$ ksi

Assuming beam dimensions of $b = 10''$ and $h_0 = 20''$
Beam self-weight = $(10 \times 20/12^2) \times 0.15 = 0.208$ k/ft
$\therefore$ Total load $w = 2 + 0.208 = 2.208$ k/ft

Maximum bending moment $M_{max} = wL^2/8 = 2.208 \times 20^2/8 = 110.4$ k-ft
$\therefore$ Assuming beam-width $b = 10''$, $d_{req} = \sqrt{(M_{max}/Rb)} = \sqrt{[110.4 \times 12/(0.223 \times 10)]} = 24.38''$
$\therefore$ Take $d_1 = 24.5''$, $h_1 = 28.5''$

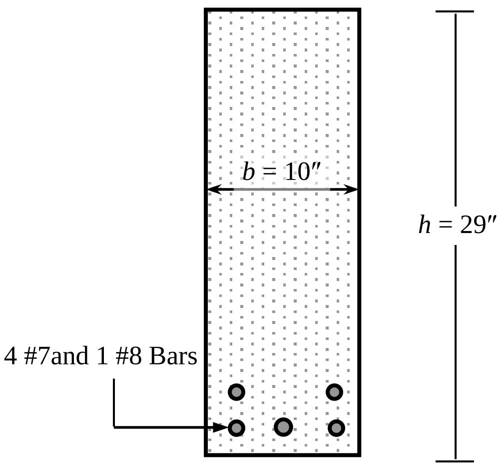
$\therefore$ Assuming beam dimensions of $b = 10''$ and $h_1 = 28.5''$
Beam self-weight = $(10 \times 28.5/12^2) \times 0.15 = 0.297$ k/ft
$\therefore$ Total load $w = 2 + 0.297 = 2.297$ k/ft

Maximum bending moment $M_{max} = wL^2/8 = 2.297 \times 20^2/8 = 114.8$ k-ft
$\therefore$ Assuming beam-width $b = 10''$, $d_{req} = \sqrt{(M_{max}/Rb)} = \sqrt{[114.8 \times 12/(0.223 \times 10)]} = 24.86''$
$\therefore$ Take $d_2 = 25.0''$, $h_2 = 29.0''$

$\therefore$ Assuming beam dimensions of $b = 10''$ and $h_2 = 29.0''$
Beam self-weight = $(10 \times 29.0/12^2) \times 0.15 = 0.302$ k/ft
$\therefore$ Total load $w = 2 + 0.302 = 2.302$ k/ft

Maximum bending moment $M_{max} = wL^2/8 = 2.302 \times 20^2/8 = 115.1$ k-ft
$\therefore$ Assuming beam-width $b = 10''$, $d_{req} = \sqrt{(M_{max}/Rb)} = \sqrt{[115.1 \times 12/(0.223 \times 10)]} = 24.89''$
$\therefore$ Take $d = 25.0''$, $h = 29.0''$, which matches with the initial assumption.

$\therefore A_s = M_{max}/(f_{sall}\, jd) = 115.1 \times 12/(20 \times 0.874 \times 25.0) = 3.16$ in^2 $\Rightarrow$ Use 4 #7 and 1#8 bars (in two layers)



Minimum Concrete Clear Cover
* 1.5″ for beams columns
* 3/4″ for slabs and walls, 2″ for footings

Minimum Distance from Bar Centers to Surface
* 2.5″ for beams columns
* 1″ for slabs and walls, 3″ for footings

Minimum Distance between Adjacent Bars
* 1″ or Bar Diameter for beams
* 1.5″ or 1.5 times Bar Diameter for columns

Combining Bar Sizes
* Preferably not more than two sizes
* Between bars of similar size (e.g., not #6 with #10)

Steel Ratio $\rho_s = A_s/bd = 3.19/(10 \times 25) = 0.0128 = 1.28 \times 10^{-2}$

Minimum Steel Ratio $\rho_{min} \cong 3\sqrt{f_c'}/f_y = 164/50000 = 3.3 \times 10^{-3}$, or often taken as $200/f_y = 0.2/50 = 4.0 \times 10^{-3}$

Working Stress Design (WSD) of Doubly Reinforced Beams

The design of singly reinforced beams assumes the concrete and steel to reach their allowable stresses simultaneously by setting an effective depth of concrete equal to $\sqrt{(M/Rb)}$ with a steel area of $M/(f_s\, jd)$ [or a steel ratio equal to the *Balanced Stress Steel Ratio* = $k/2r$].

However, it is often impractical or uneconomical to design RC beams as only singly reinforced beams, mainly because

* It is sometimes necessary to limit the depth of beams for architectural or service requirements, so that concrete itself may be insufficient to withstand compression and need additional steel in the compression zone
* Since beams need to be reinforced at both the top and bottom (to facilitate placement of shear reinforcements), it is uneconomical to ignore the effect of compression reinforcements in designing them. Therefore it is more rational to design beams as doubly reinforced, utilizing contribution of reinforcements in compression as well as tension.

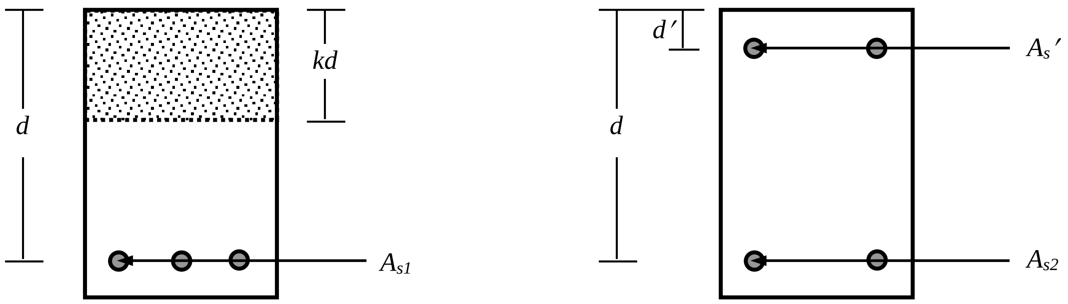


Fig. 5.1: Doubly Reinforced Beam resisting applied moments (i) M_1 (= M_c), (ii) M_2 (= $M - M_c$)

The design of doubly reinforced beams is divided into two parts, considering moments to be resisted by the compressive stress of concrete and compression reinforcements. Since the effective depth $d < d_{(req)}$; i.e., the applied moment $M > M_c$ (= Rbd^2). This moment is divided into two parts; i.e., $M_1 = M_c$ and $M_2 = M - M_c$.

The moment M_1 is given by

$$M_1 = Rbd^2 = A_{s1}\,(f_s\, jd) \qquad \text{………………. (5.1)}$$

$\therefore$ The required steel area (A_{s1}) resisting the moment M_1 is given by

$$A_{s1} = M_1/(f_s\, jd) \qquad \text{………………. (5.2)}$$

while the remaining moment M_2 is resisted by additional tensile steel (area A_{s2}) and compression steel (area A_s'); i.e.,

$$M_2 = A_{s2}\, f_s\,(d-d') = A_s' f_s'\,(d-d') \qquad \text{………………. (5.3)}$$

where d' is the depth of compression steel from the compression edge of the beam.

$\therefore$ The required steel areas resisting the moment M_2 are given by

The additional tensile reinforcement $\quad A_{s2} = M_2/[f_s\,(d-d')]$ ………………. (5.4)

i.e., The total tensile reinforcement $\quad A_s = A_{s1} + A_{s2} = M_1/(f_s\, jd) + M_2/[f_s\,(d-d')]$ ………………. (5.5)

while the compression reinforcement $\quad A_s' = M_2/[f_s'\,(d-d')]$ ………………. (5.6)

The compressive stress f_s' can be evaluated based on the strain compatibility; i.e.,

$$\varepsilon_s/(d-kd) = \varepsilon_s'/(kd-d') \Rightarrow f_s' = f_s\,(k-d'/d)/(1-k) \qquad \text{………………. (5.7)}$$

Eq. (5.7) is based on completely elastic behavior of concrete and steel in compression. However, this assumption is not valid at higher strains, when stresses no longer vary proportionately with strain. Since the strains in compression steel and adjacent concrete remain equal, this means that at higher strain levels, concrete 'transfers' part of the compressive stress to steel, and the stress in steel, being proportional to strain will be larger than it would be if the concrete behaved elastically. This is accentuated by the fact that concrete compresses under constant load or stress (flow or creep).

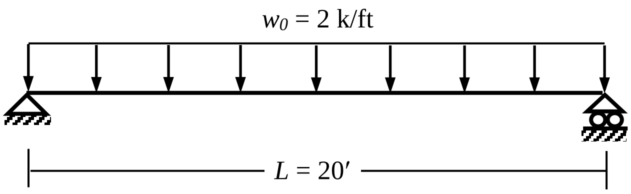
To approximate the effects of nonlinear stress-strain behavior and plastic flow of concrete, ACI code provisions specify that effective modular ratio of $2E_s/E_c$ be used to transform the steel area to concrete, thereby doubling the stress computed in Eq. (5.7), which of course should not exceed the allowable stress in tension.

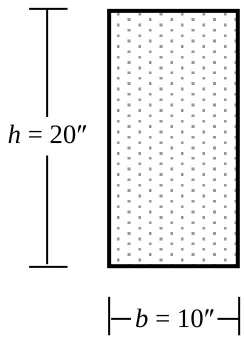
Therefore, Eq. (5.7) takes the ACI code-based form

$$f_s' = 2f_s(k - d'/d)/(1-k) \le f_s \qquad \text{………………. (5.8)}$$

Example 5.1

Use the WSD Method to design the simply supported RC beam loaded as shown below, in addition to its self-weight [Given: $f_c' = 3$ ksi, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi].





For $f_c' = 3$ ksi = 3000 psi, $n = 9$, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi $\Rightarrow r = 14.81$

$\therefore k = 0.378 \Rightarrow j = 1 - k/3 = 0.874$, $R = 0.5\, f_{call}\, kj = 0.223$ ksi

Given beam dimensions of $b = 10''$ and $h = 20''$

Beam self-weight = $(10 \times 20/12^2) \times 0.15 = 0.208$ k/ft $\Rightarrow$ Total load $w = 2 + 0.208 = 2.208$ k/ft

Maximum bending moment $M_{max} = wL^2/8 = 2.208 \times 20^2/8 = 110.4$ k-ft

$\therefore$ Assuming two layers of bottom steel, $d = 20 - 4 = 16''$

$\therefore M_1 = Rbd^2 = 0.223 \times 10 \times 16^2/12 = 47.56$ k-ft $\Rightarrow M_2 = M - M_1 = 62.85$ k-ft

$\therefore A_s = A_{s1} + A_{s2} = M_1/(f_s jd) + M_2/[f_s(d-d')] = 47.56 \times 12/(20 \times 0.874 \times 16) + 62.85 \times 12/[20 \times (16 - 2.5)]$

$= 2.04 + 2.79 = 4.83$ in^2; i.e., 6 #8 Bars in two layers

Also $f_s' = 2f_s(k - d'/d)/(1-k) = 2 \times 20 \times (0.378 - 2.5/16)/(1-0.378) = 14.25$ ksi, which is < 20 ksi

$\Rightarrow A_s' = M_2/[f_s'(d-d')] = 62.85 \times 12/[14.25 \times (16 - 2.5)] = 3.92$ in^2; i.e., Difficult to place in one layer

$\therefore$ Two layers of steel $\Rightarrow d' = 4''$

$\therefore A_s = A_{s1} + A_{s2} = M_1/(f_s jd) + M_2/[f_s(d-d')] = 2.04 + 3.14 = 5.18$ in^2

$f_s' = 2f_s(k - d'/d)/(1-k) = 2 \times 20 \times (0.378 - 4/16)/(1-0.378) = 8.23$ ksi, which is < 20 ksi

$\Rightarrow A_s' = 62.85 \times 12/[8.23 \times (16 - 4)] = 7.64$ in^2; which is difficult to place in two layers.

$\therefore$ This depth is too small for the beam.

The following cross-sections are suggested are alternative.

Assuming $b = 10''$ and $h = 24''$

Beam self-weight = $(10 \times 24/12^2) \times 0.15 = 0.25$ k/ft

$\therefore$ Total load $w = 2 + 0.25 = 2.25$ k/ft

$M_{max} = wL^2/8 = 2.25 \times 20^2/8 = 112.5$ k-ft

$\therefore$ Assuming two layers of steel, $d = 24 - 4 = 20''$

$\therefore M_1 = Rbd^2 = 0.223 \times 10 \times 20^2/12 = 74.32$ k-ft

$\Rightarrow M_2 = M - M_1 = 38.18$ k-ft

$\therefore A_s = M_1/(f_s jd) + M_2/[f_s(d-d')]$

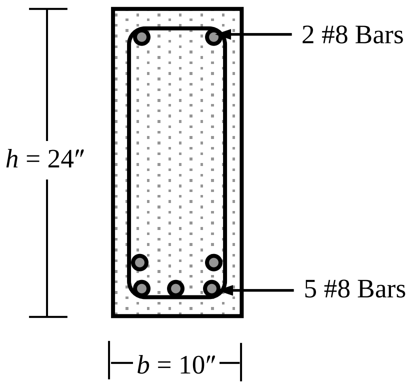
$= 74.32 \times 12/(20 \times 0.874 \times 20) + 38.18 \times 12/[20 \times 17.5]$

$= 2.55 + 1.31 = 3.86$ in^2; i.e., 5 #8 Bars in two layers

Also $f_s' = 2 \times 20 \times (0.378 - 2.5/20)/(1-0.378) = 16.26$ ksi

$\Rightarrow A_s' = 38.18 \times 12/[16.26 \times (20-2.5)] = 1.61$ in^2;

i.e., 2 #8 Bars in one layer



Assuming $b = 12''$ and $h = 20''$

Beam self-weight = $(12 \times 20/12^2) \times 0.15 = 0.25$ k/ft

$\therefore$ Total load $w = 2 + 0.25 = 2.25$ k/ft

$M_{max} = wL^2/8 = 2.25 \times 20^2/8 = 112.5$ k-ft

$\therefore$ Assuming two layers of steel, $d = 20 - 4 = 16''$

$\therefore M_1 = Rbd^2 = 0.223 \times 12 \times 16^2/12 = 57.08$ k-ft

$\Rightarrow M_2 = M - M_1 = 55.42$ k-ft

$\therefore A_s = M_1/(f_s jd) + M_2/[f_s(d-d')]$

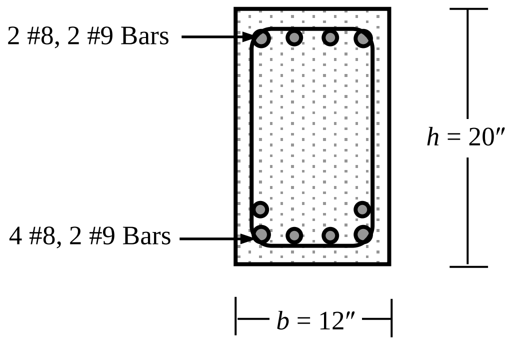
$= 57.08 \times 12/(20 \times 0.874 \times 16) + 55.42 \times 12/[20 \times 13.5]$

$= 2.45 + 2.46 = 4.91$ in^2; i.e., 4 #8, 2 #9 Bars in two layers

Also $f_s' = 2 \times 20 \times (0.378 - 2.5/16)/(1-0.378) = 14.25$ ksi

$\Rightarrow A_s' = 55.42 \times 12/[14.25 \times (16-2.5)] = 3.46$ in^2;

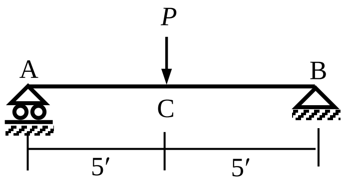
i.e., 2 #8, 2 #9 Bars in one layer

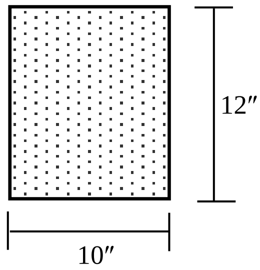


Questions and Problems (2)

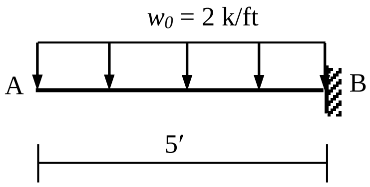
1. (i) What is the Working Stress Design (WSD) of Reinforced Concrete? Mention the main features of WSD.
   (ii) Show the variations of stress and strain over an RC section as it is stressed gradually from uncracked to cracked and ultimate failure condition.
   (iii) What is the balanced stress steel ratio and minimum steel ratio used in RC design? Explain why they are used.
   (iv) Why does the ACI recommend that in WSD, the value of compressive stress in steel (f_s') be taken as twice the value calculated from elastic analysis?

2. Use the WSD method to design the section C of the simply supported RC beam AB shown in the figures below if (i) $P = 0$, (ii) $P = 5$ kips, (iii) $P = 10$ kips. Include the self-weight of the section also [Given: $f_c' = 3$ ksi, $f_s = 18$ ksi].



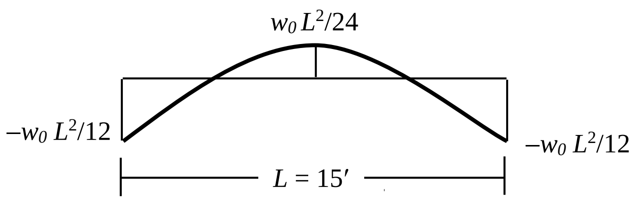


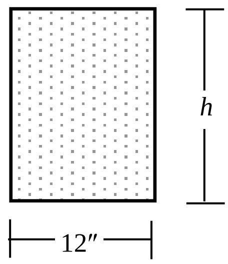
3. Use the WSD method to design section B of the cantilever RC beam AB shown in the figure below as a singly reinforced beam (i) excluding its self-weight, (ii) including its self-weight [Given: $f_c' = 4$ ksi, $f_s = 24$ ksi].



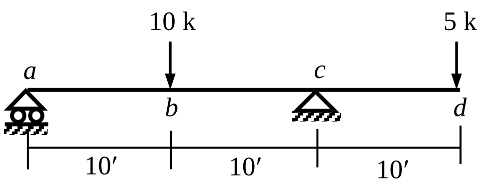
4. Design the cantilever beam shown in Question 3 if it has a (10″ × 12″) cross-section shown in Question 2.

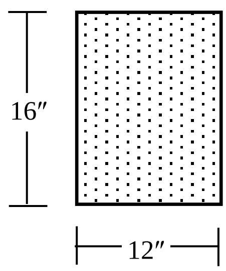
5. The figures below show the bending moment diagram and cross-section of a Reinforced Concrete beam.
   If $w_0 = 2$ k/ft, design the beam as a
   (i) singly reinforced beam for the maximum positive moment,
   (ii) doubly reinforced beam for the negative moment, using the depth h calculated in (i)
   [Given: $f_c' = 4$ ksi, $f_{sall} = 20$ ksi].





6. Design the RC beam *abcd* loaded as shown below (in addition to self weight) for the maximum positive and negative moments, if it has a (12″ × 16″) cross-section shown alongside [Given: $f_c' = 3$ ksi, $f_s = 20$ ksi].





Design Concepts of Ultimate Strength Design (USD)

The Working Stress Design (WSD) method designs RC sections assuming them to be within their elastic limits, where stresses are proportional to strains. Large margins or factors of safety are assumed on material strengths to ensure such behavior. It is equally, if not more important to predict the ultimate strength of RC sections so that they can be designed to resist the largest loads anticipated during their design lives. The materials are not expected to remain within their elastic limits at such high stresses. More realistic methods of analysis, based on actual inelastic behavior rather than assumed elastic behavior of materials and on results of extremely extensive experimental research, have been performed to predict the ultimate strengths. The Ultimate Strength Design (USD) method, derived from such works, is now used extensively (and almost exclusively in many countries) in structural design practice.

Fig. 4.1 (repeated and slightly modified) shows the variation of strain and stress of a rectangular RC section subjected to increasing positive bending moments, demonstrating (i) uncracked section (upto tensile cracking of concrete), (ii) cracked elastic section (used in WSD), (iii) cracked inelastic section (beyond elastic limit of concrete, used in USD).

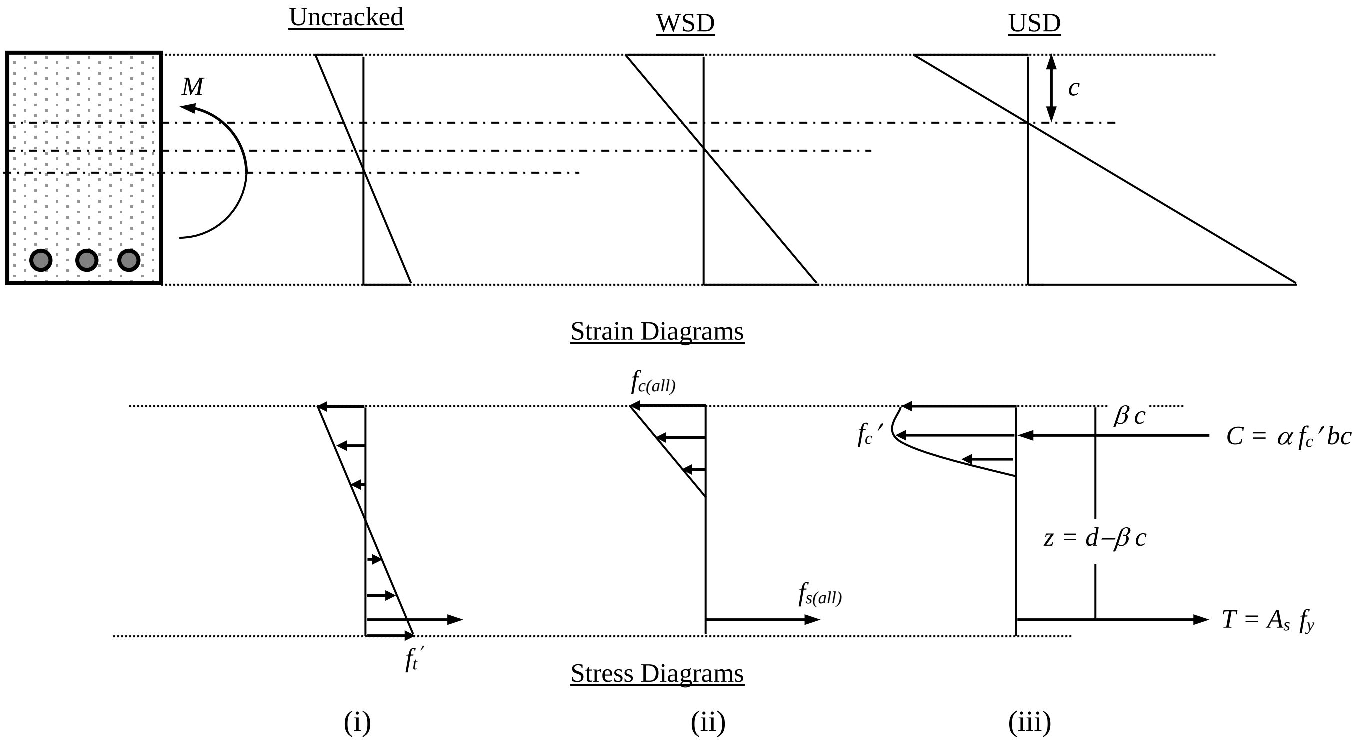


Fig. 4.1 (Repeated and modified): Strain and Stress Diagram of RC sections (i) Uncracked, (ii) WSD, (iii) USD

As shown in Fig. 4.1 (iii), in a rectangular beam the area in compression is bc, and the total compressive force on this area can be expressed as $C = \alpha f_c' bc$, where $\alpha f_c'$ (= f_{cav}) is the average compressive stress on the area bc. For a given distance 'c' to the neutral axis, the location of C can be defined as some fraction β of this distance.

Therefore, the ultimate strength of the section can be calculated from equilibrium and the following equations

$C = T \Rightarrow \alpha f_c' bc = A_s f_y \Rightarrow c = A_s f_y / \alpha f_c' b$ ……………….. (6.1)

$M_c = C\,z = \alpha f_c' bc\,(d - \beta c)$ ……………….. (6.2)

$M_s = T\,z = A_s f_y\,(d - \beta c)$ ……………….. (6.3)

Extensive direct measurements as well as indirect evaluation of numerous beam tests have given the following values of α and β for various strengths of concrete (varies due to the differing strain-stress curves; i.e., more brittle high strength concretes) shown in Table 6.1.

Table 6.1: Variation of α and β with concrete strength (f_c')

f_c'(ksi)	≤ 4	4~8	≥ 8
α	0.72	$0.72 - 0.04\,(f_c' - 4)$	0.56
β	0.425	$0.425 - 0.025\,(f_c' - 4)$	0.325

Failure Modes and Balanced Steel Ratio

Eqs. (6.1~3) were derived on the assumption of *Tension Failure*; i.e., that steel reinforcements would yield ($f_s = f_y$) as concrete reaches its ultimate crushing strength f_c'. These equations can be combined to a single equation for the nominal moment capacity of the section

$$M_n = A_s f_y (d - \beta A_s f_y / \alpha f_c' b) \qquad \text{.................... (6.4a)}$$

$$= \rho_s f_y (1 - \beta \rho_s f_y / \alpha f_c') bd^2 = \rho_s f_y (1 - 0.59 \rho_s f_y / f_c') bd^2 \qquad \text{.................... (6.4b)}$$

where $\rho_s = A_s/bd$ is the steel ratio of the beam-section.

However, if the steel ratio is too high, the reinforcements may not yield when the concrete reaches its capacity. This failure mode, i.e., the *Compression Failure* occurs when the compression strain in concrete reaches its ultimate strain (ε_u = 0.003) before the yielding of steel ($f_s < f_y$). The steel stress is then proportional to the strain, and is given by

$$f_s = E_s \varepsilon_s \qquad \text{.................... (6.5)}$$

The steel strain can be obtained from the strain distribution, using similar triangles; i.e., $\varepsilon_s = \varepsilon_u (d - c)/c$ .... (6.6)

And then, using $C = T \Rightarrow \alpha f_c' bc = A_s f_s = A_s E_s \varepsilon_u (d - c)/c$ .... (6.7)

Combining Eq. (6.5~6.7), the depth c is obtained as

$$c = d [-(m\rho_s/2) + \sqrt{\{(m\rho_s/2)^2 + m\rho_s\}}] \qquad \text{................ (6.8)}$$

where, $m = E_s \varepsilon_u / \alpha f_c'$, and ρ_s is the steel ratio.

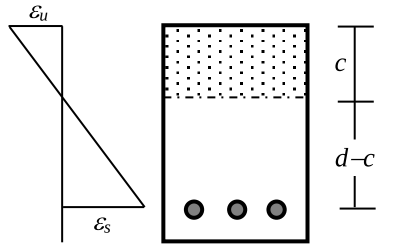


Fig. 6.1: Strains at Compression Failure

Once c is obtained, Eq. (6.2) [$M_c = \alpha f_c' bc (d - \beta c)$] can be used to calculate the nominal moment capacity M_n.

Between the two modes, compression failure occurs explosively and without any warning of distress. For this reason, it is good practice to keep the amount of reinforcement sufficiently small to ensure that, should the member be overstressed, it will give adequate warning before failing in a gradual manner by yielding of steel (accompanied by deflections and widening of concrete cracks) rather than by crushing of concrete. This can be done by keeping the reinforcement ratio below a certain limiting value. This value, the so-called *Balanced Steel Ratio* (ρ_b), represents the amount of reinforcement necessary to make the beam fail by crushing of concrete at the same load that causes the yielding of steel. Therefore, in this case, Eq. (6.6) can be modified as

$$\varepsilon_y = \varepsilon_u (d - c)/c \Rightarrow c = \{\varepsilon_u/(\varepsilon_u + \varepsilon_y)\} d \qquad \text{.................... (6.9)}$$

$$\therefore \text{Eq. (6.1)} \Rightarrow c = A_s f_y / \alpha f_c' b \Rightarrow \rho_b = (\alpha f_c'/f_y)\{\varepsilon_u/(\varepsilon_u + \varepsilon_y)\} \qquad \text{................ (6.10a)}$$

Taking ε_u = 0.003 and $\varepsilon_y = f_y$ / 29000, ρ_b can be modified as = $(\alpha f_c'/f_y)$ {87/(87 + f_y)} .... (6.10b)

For practical purpose, beams are seldom designed for steel ratios as high as the *Balanced Steel Ratio* (ρ_b). Since steel strains very little before yield, and to consider uncertainties in material properties, strain-hardening and actual A_s, ACI recommends a maximum steel ratio of $\rho_{max} = 0.75\rho_b$ in structural design to ensure ductile failure of beams.

Equivalent Rectangular Stress Distribution

It was earlier noted that the actual geometric shape of the concrete compressive-stress distribution varies considerably and needs not be known for design purpose provided two things are known: (i) magnitude of the resultant compressive force C (defined by the factor α), and (ii) location of the resultant (defined by β). Therefore, it is convenient if the actual complex stress distribution can be replaced by a fictitious 'equivalent' stress distribution keeping the resultant and its location the same. Among several simplified stress distributions, the one proposed by C. S. Whitney (assuming rectangular stress distribution) is the most widely used.

Analysis and Design of Singly Reinforced Beam

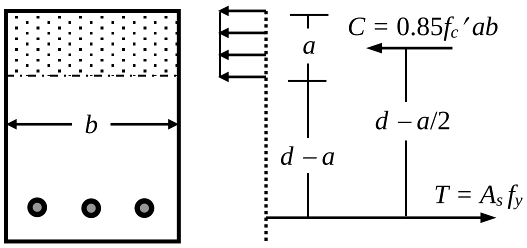


Fig. 6.2: Forces from Rectangular Stress Block

Compressive force in concrete, $C = 0.85 f_c' ab$ .... (6.11)

Tensile force in steel, $T = A_s f_s = A_s f_y$ .... (6.12)
[assuming yielding of steel]

$\therefore$ Equating the two, $0.85 f_c' ab = A_s f_y$

$\Rightarrow$ Depth of rectangular stress block, $a = A_s f_y / (0.85 f_c' b)$ ....(6.13)

$\therefore$ Nominal moment capacity, $M_n = A_s f_y (d - a/2)$ ....(6.14)

Combining Eqs. (6.13), (6.14) and taking $f_c = 0.85 f_c' \Rightarrow a = d [1 - \sqrt{\{1 - 2M_n/(f_c bd^2)\}}]$ .... (6.15)

Also (6.13) $\Rightarrow A_s = (f_c/f_y) [1 - \sqrt{\{1 - 2 M_n/(f_c bd^2)\}}] bd$ .... (6.16)

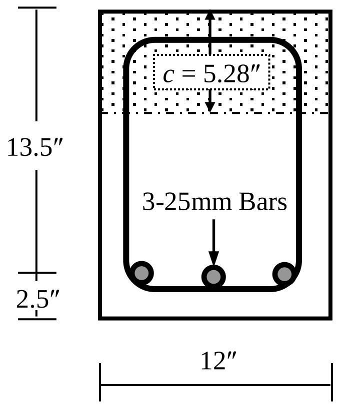
Analysis of Singly Reinforced Beam

Example 6.1

Calculate the nominal moment capacity of the singly reinforced beam sections [Given: $f_c' = 3$ ksi, $f_y = 60$ ksi].

For $f_c' = 3$ ksi, $\alpha = 0.72$, $\beta = 0.425 \Rightarrow$ Balanced Steel Ratio, $\rho_b = (0.72 \times 3/60)\{87/(87+60)\} = 0.0213$

$\Rightarrow A_{sb} = 0.0213 \times 12 \times 13.5 = 3.45$ in^2



Section 1

In Section 1, $\rho_s = A_s/bd = 2.28/162 = 0.0141 < \rho_b$

$\therefore c = A_s f_y / \alpha f_c' b = 2.28 \times 60/(0.72 \times 3 \times 12) = 5.28''$

$M_n = A_s f_y (d - \beta c) = 2.28 \times 60\ (13.5 - 0.425 \times 5.28)/12$
$= 128.3$ k′

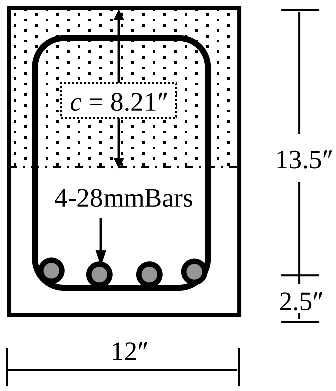
In Section 2, $\rho_s = A_s/bd = 4.00/162 = 0.0235 > \rho_b$

$\therefore m = E_s \varepsilon_u / \alpha f_c' = 87/(0.72 \times 3) = 40.28$

$m\rho_s = 40.28 \times 0.0235 = 0.945$

$c = d\,[-(m\rho_s/2) + \sqrt{\{(m\rho_s/2)^2 + m\rho_s\}}]$
$= 13.5\,[-0.472 + \sqrt{(0.472^2 + 0.945)}] = 8.21''$

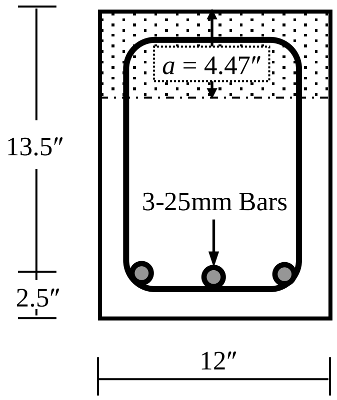
$M_n = \alpha f_c' bc\,(d - \beta c)$
$= 0.72 \times 3 \times 12 \times 8.21\ (13.5 - 0.425 \times 8.21)/12 = 177.6$ k′
[where $f_s = 87\,(d-c)/c = 87 \times 5.29/8.21 = 56.02$ ksi]



Section 2

Example 6.2

Repeat Example 6.1 using the Rectangular Stress Block.



Section 1

In Section 1, $\rho_s = A_s/bd = 2.28/162 = 0.0141 < \rho_b$

$\therefore a = A_s f_y/(0.85 f_c' b) = 2.28 \times 60/(0.85 \times 3 \times 12) = 4.47''$

$M_n = A_s f_y\,(d - a/2) = 2.28 \times 60\ (13.5 - 4.47/2)/12$
$= 128.4$ k′

OR

$M_n = \rho_s f_y\,(1 - 0.59\,\rho_s f_y / f_c')\,bd^2$
$= 0.844\,(1 - 0.59 \times 0.844/3) \times 12 \times 13.5^2/12 = 128.3$ k′

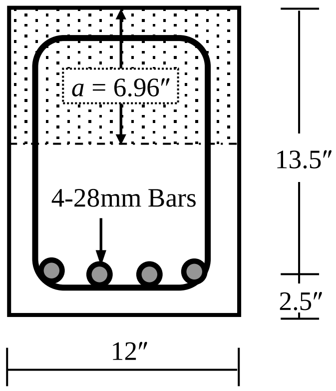
In Section 2, $\rho_s = A_s/bd = 3.80/162 = 0.0235 > \rho_b$

$\therefore m = 40.28$, $m\rho_s = 0.945$

$c = d\,[-(m\rho_s/2) + \sqrt{\{(m\rho_s/2)^2 + m\rho_s\}}] = 8.21''$

$\therefore a = (0.72/0.85) \times 8.21 = 6.96''$

$\therefore M_n = 0.85\,f_c' ab\,(d - a/2)$
$= 0.85 \times 3 \times 6.96 \times 12\ (13.5 - 6.96/2)/12 = 177.8$ k′



Section 2

Load and Resistance Factors

In order to account for the uncertainty in environmental loads, they are increased by overload factors (> 1.0) in calculating the applied internal forces (i.e., moments, shears and axial forces). In addition, the nominal resistance capacities of sections are reduced by resistance factors ϕ (< 1.0). The design criteria in USD are therefore set as

$$M_u < \phi M_n, \qquad V_u < \phi V_n, \qquad P_u < \phi P_n \qquad \text{.......................................(6.17)}$$

According to ACI, the overload factors for dead load, live load, wind and earthquake can be set as 1.4, 1.7, 1.7 and 1.87, while $\phi = 0.90$, $\phi = 0.85$ for shear, and $\phi = 0.70$ or 0.75 for axial forces.

Example 6.3

Calculate the ultimate moment capacity of the sections shown in Example 6.1 and 6.2.

The nominal moment capacities are calculated as $M_n = 128.4$ k′, and $M_n = 177.8$ k′

Therefore the ultimate design moment capacities are

$M_u = 0.90 \times 128.4 = 115.6$ k′, and $M_n = 0.90 \times 177.8 = 160.0$ k′

Example 6.4

For a beam section (with beam width $b = 12''$) subject to applied ultimate moment $M_u = 90$ k-ft, calculate the
(i) minimum depth of the section and steel area assuming steel ratio to be equal to ρ_{max}
(ii) steel reinforcement if the beam height (h) is fixed at 16″ [Given: $f_c' = 3$ ksi, $f_y = 60$ ksi].

For $f_c' = 3$ ksi, $\alpha = 0.72$, $\beta = 0.425 \Rightarrow$ Maximum Steel Ratio, $\rho_{max} = 0.75\ (0.72 \times 3/60)\ \{87/(87 + 60)\} = 0.0160$

Applied ultimate moment $M_u = 90$ k-ft $\Rightarrow$ Nominal moment capacity $M_n = 90/0.90 = 100$ k-ft

(i) Using Eqs. (6.4), $M_n = \rho_s f_y\ (1 - 0.59\ \rho_s f_y/f_c')\ bd^2$

$\Rightarrow 100 \times 12 = 0.0160 \times 60\ (1 - 0.59 \times 0.0160 \times 60/3) \times 12 \times d^2 = (0.778)\ 12 \times d^2 \Rightarrow d_{req} = 11.34''$

$\Rightarrow A_s = 0.0160 \times 12 \times 11.34 = 2.17$ in^2; $\therefore$ Use 2 #8, 1 #7 Bars

Also take $d_{req} = 11.5'' \Rightarrow h = 14''$ (assuming one layer of steel)

(ii) $h = 16'' \Rightarrow d = 13.5''$ (assuming one layer of steel)

Assuming $a = 3'' \Rightarrow A_s = M_n/\{f_y\ (d - a/2)\} = 100 \times 12/\{60 \times (13.5 - 3/2)\} = 1.67$ in^2

$\Rightarrow a = A_s\ f_y/(0.85 f_c' b) = 1.67 \times 60/(0.85 \times 3 \times 12) = 3.27''$

$\therefore a = 3.27'' \Rightarrow A_s = 100 \times 12/\{60 \times (13.5 - 3.27/2)\} = 1.69$ in$^2 \Rightarrow a = 1.69 \times 60/(0.85 \times 3 \times 12) = 3.31''$

$a = 3.31'' \Rightarrow A_s = 100 \times 12/\{60 \times (13.5 - 3.31/2)\} = 1.69$ in^2; $\therefore$ Use 4 #6 Bars

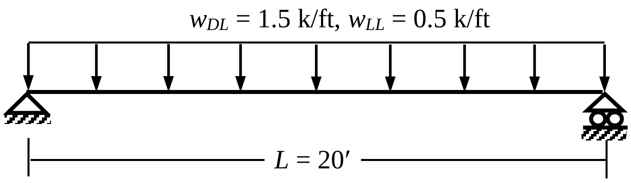
Alternatively, with $f_c = 0.85 f_c' = 2.55$ ksi, and $M_n/(f_c\ bd^2) = 100 \times 12/(2.55 \times 12 \times 13.5^2) = 0.215$

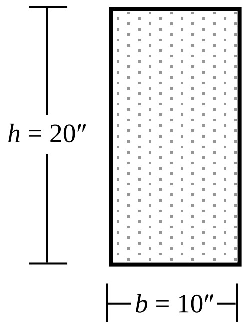
$a = d\ [1 - \sqrt{\{1 - 2\ M_n/(f_c bd^2)\}}] = 13.5\ [1 - \sqrt{\{1 - 2 \times 0.215\}}] = 3.31''$

$\Rightarrow A_s = (0.85 f_c' ab)/f_y = 2.55 \times 3.31 \times 12/60 = 1.69$ in^2

Example 6.5

Use the USD Method to design the simply supported RC beam loaded as shown below, in addition to its self-weight [Given: $f_c' = 3$ ksi, $f_y = 50$ ksi].





Given beam dimensions of $b = 10''$ and $h = 20''$

Beam self-weight = $(10 \times 20/12^2) \times 0.15 = 0.208$ k/ft

$\Rightarrow$ Total ultimate load $w_u = 1.4 \times 0.208 + 1.4 \times 1.5 + 1.7 \times 0.5 = 3.242$ k/ft

$\therefore$ Applied maximum moment $M_u = w_u L^2/8 = 162.1$ k-ft; i.e., Nominal moment $M_n = M_u/\phi = 180.1$ k-ft

$h = 20'' \Rightarrow d = 16''$ (assuming two layers of steel), $M_n/(f_c\ bd^2) = 180.1 \times 12/(2.55 \times 10 \times 16^2) = 0.331$

$\therefore a = d\ [1 - \sqrt{\{1 - 2\ M_n/(f_c bd^2)\}}] = 16\ [1 - \sqrt{\{1 - 2 \times 0.331\}}] = 6.70''$

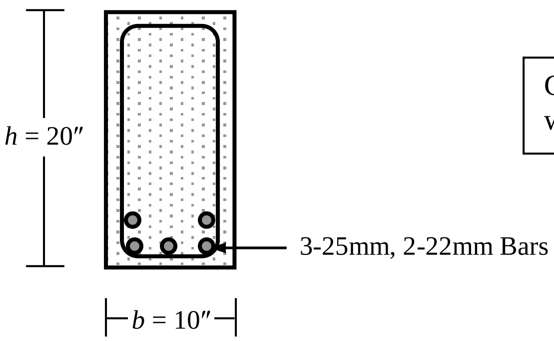
$\Rightarrow A_s = (0.85 f_c' ab)/f_y = 2.55 \times 6.70 \times 10/50 = 3.42$ in^2

Steel Ratio $\rho_s = A_s/bd = 3.42/(10 \times 16) = 0.0214$

while $\rho_b = (0.72 \times 3/50)\ \{87/(87 + 50)\} = 0.0274 \Rightarrow \rho_{bmax} = 0.75 \rho_b = 0.0206$

$\therefore$ Steel ratio is less than ρ_b but marginally greater than ρ_{bmax}

i.e., The section may be considered OK, particulary taking the compression reinforcements into account.



Compare the section (and reinforcements) with the one obtained from WSD

Analysis of Doubly Reinforced Beam

Because of the high design stresses, Doubly Reinforced beams are not as common as they are in WSD. But they are useful for control of crack and deflection as well as binding the top and bottom reinforcements.

Moment Resisting Mechanism for Doubly Reinforced Beam

As shown in the formulation for WSD, in doubly reinforced beam, the nominal moment is resisted by two mechanisms; i.e., reinforcement A_{s1} acting with the concrete in compression, and reinforcement A_{s2} (= $A_s - A_{s1}$) acting with the compression reinforcement A_s'. From equilibrium, these two forces are given by

$A_{s1} f_y = 0.85 f_c' ab \Rightarrow a = A_{s1} f_y/(0.85 f_c' b)$ …..…………(6.18)

and $A_{s2} f_y = A_s' f_s' \Rightarrow A_{s2} = A_s' f_s'/ f_y$ …..…………(6.19)

The value of f_s' can be calculated from the strain compatibility equation $\varepsilon_s'/\varepsilon_u = (c - d')/c$

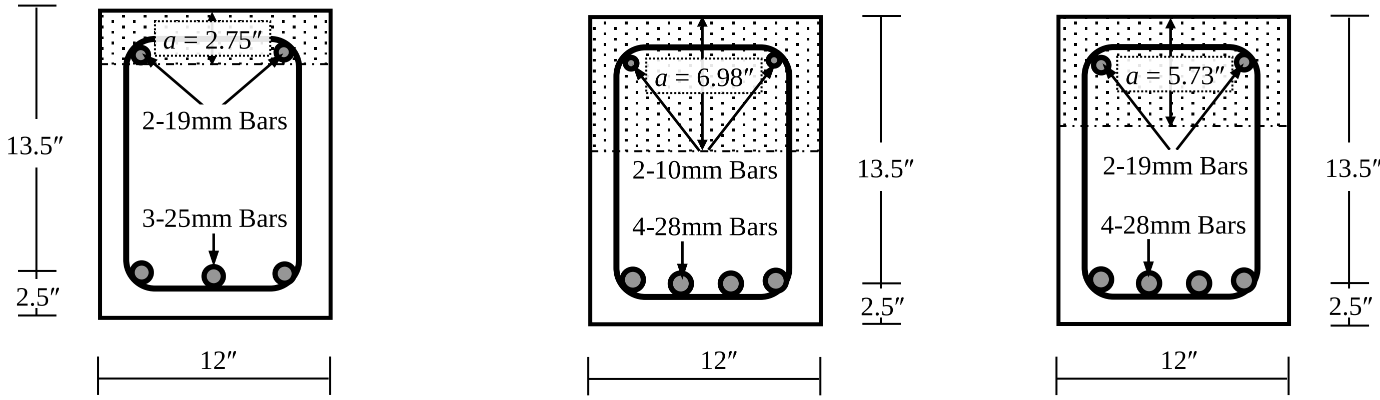
$\Rightarrow f_s' = E_s \varepsilon_u\ (c - d')/c \leq f_y$ ……………(6.20)

From which the ultimate moment capacity is calculated to be $M_u = \phi A_{s1} f_y\ (d - a/2) + \phi A_{s2} f_y\ (d - d')$ ……………(6.21)

If both the tensile and compression reinforcements yield, $M_u = \phi (A_s - A_s') f_y\ (d - a/2) + \phi A_s' f_y\ (d - d')$ …..……(6.22)

Example 6.6

Calculate the ultimate moment capacities of the RC sections shown below [Given: f_c' = 3 ksi, f_y = 60 ksi].



Section 1 **Section 2** **Section 3**

In Section 1, $\rho_s = A_s/bd$ = 2.28/162 = 0.0141 < ρ_b, and $\rho_s' = A_s'/bd$ = 0.88/162 = 0.0054

$\therefore a = (A_s - A_s') f_y/(0.85 f_c' b)$ = 1.40 × 60/(0.85 × 3 × 12) = 2.75″ $\Rightarrow c = (0.85/\alpha)\ a$ = 3.24″

Actual stress in compression rod, f_s' = 87 $(c - d')/c$ = 87 (3.24– 2.5)/3.24 = 19.9 ksi < 60 ksi

Effective $A_{s2} = A_s' f_s'/ f_y$ = 0.88 × 19.9/60 = 0.29 in^2 $\Rightarrow A_{s1}$ = 2.28 – 0.29 = 1.99 in^2

$\Rightarrow a$ = 1.99 × 60/(0.85 × 3 × 12) = 3.90″

$M_u = \phi A_{s1} f_y\ (d - a/2) + \phi A_{s2} f_y\ (d - d')$ = 0.90 × {1.99 × 60 (13.5 – 3.90/2) + 0.29 × 60 (13.5 – 2.5)}/12 = 117.8 k-ft

$\therefore$ Effect of A_s' can be neglected, approximating M_u by 115.6 k-ft (calculated for singly reinforced beam)

In Section 2, $\rho_s = A_s/bd$ = 3.80/162 = 0.0235, and $\rho_s' = A_s'/bd$ = 0.24/162 = 0.0015

$\therefore \rho_s > \rho_b$ (= 0.0213), and $(\rho_s - \rho_s')$ = 0.0220 > ρ_b

$\therefore a = (A_s - A_s') f_y/(0.85 f_c' b)$ = 3.56 × 60/(0.85 × 3 × 12) = 6.98″ $\Rightarrow c = (0.85/\alpha)\ a$ = 8.24″

Actual stress in compression rod, f_s' = 87 $(c - d')/c$ = 87 (8.24– 2.5)/8.24 = 60.6 ksi > 60 ksi $\Rightarrow f_s'$ = 60 ksi

$M_u = \phi (A_s - A_s') f_y\ (d - a/2) + \phi A_s' f_y\ (d - d')$ = 0.90 × {3.56 × 60 (13.5 – 6.98/2) + 0.24 × 60 (13.5 – 2.5)}/12
= 160.4 + 11.9 = 172.2 k-ft; i.e., Significantly greater than 160.1 k-ft (calculated for singly reinforced beam)

In Section 3, $\rho_s = A_s/bd$ = 3.80/162 = 0.0235 > ρ_b, and $\rho_s' = A_s'/bd$ = 0.88/162 = 0.0054

$\therefore a = (A_s - A_s') f_y/(0.85 f_c' b)$ = 2.92 × 60/(0.85 × 3 × 12) = 5.73″ $\Rightarrow c = (0.85/\alpha)\ a$ = 6.76″

Actual stress in compression rod, f_s' = 87 $(c - d')/c$ = 87 (6.76– 2.5)/6.76 = 54.8 ksi < 60 ksi

Effective $A_{s2} = A_s' f_s'/ f_y$ = 0.88 × 54.8/60 = 0.80 in^2 $\Rightarrow A_{s1}$ = 3.80 – 0.80 = 3.00 in^2

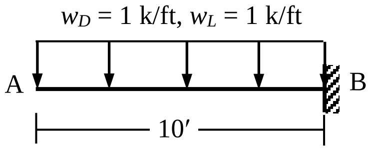
$\Rightarrow a$ = 3.00 × 60/(0.85 × 3 × 12) = 5.73″

$M_u = \phi A_{s1} f_y\ (d - a/2) + \phi A_{s2} f_y\ (d - d')$ = 0.90 × {3.00 × 60 (13.5– 5.73/2) + 0.80 × 60 (13.5 – 2.5)}/12 = 183.2 k-ft; i.e., Also significantly greater than 160.1 k-ft (calculated for singly reinforced beam)

USD of Doubly Reinforced Beam

Example 6.7

Use the USD to design the cantilever beam AB loaded as shown in the figure below (in addition to its self-weight), if working dead load w_D = 1 k/ft, and working live load w_L = 1 k/ft, assuming a steel ratio of ρ_{max},
(i) as singly reinforced beam (b = 12″), (ii) (12″ × 14″) cross-section [Given: f_c' = 4 ksi, f_s = 60 ksi].



Using load factors 1.4 for dead load and 1.7 for live load, the ultimate load, $w_u = 1.4 \times 1 + 1.7 \times 1 = 3.1$ k/ft
$\therefore$ Ultimate moment, $M_u = 3.1 \times 10^2/2 = 155.0$ k-ft $\Rightarrow M_n = 155.0/0.9 = 172.2$ k-ft

For f_c' = 4 ksi, $\alpha = 0.72 \Rightarrow$ Balanced Steel Ratio, $\rho_b = (0.72 \times 4/60)\ \{87/(87 + 60)\} = 0.0284$
$\therefore$ Steel Ratio, $\rho_s = 0.75 \times 0.0284 = 0.0213$

(i) Using Eqs. (6.4), $M_n = \rho_s f_y (1 - 0.59\ \rho_s f_y/f_c')\ bd^2$
$\Rightarrow 172.2 \times 12 = 0.0213 \times 60\ (1 - 0.59 \times 0.0213 \times 60/4) \times 12 \times d^2 = (1.037) \times 12 \times d^2$
$\Rightarrow d_{req} = 12.89''$
$\therefore d = 13.0'' \Rightarrow$ Cross-section (12″ × 15.5″) $\Rightarrow$ Self weight $= 1.4 \times (12 \times 15.5)/12^2 \times 0.15 = 0.27$ k/ft

$\therefore$ Ultimate moment, $M_u = (3.1 + 0.27) \times 10^2/2 = 168.6$ k-ft $\Rightarrow M_n = 168.6/0.9 = 187.3$ k-ft
$\Rightarrow 187.3 \times 12 = (1.037) \times 12 \times d^2 \Rightarrow d_{req} = 13.44''$
$\Rightarrow A_s = 0.0213 \times 12 \times 13.44 = 3.44$ in^2; $\therefore$ Use 2#8, 2#9 Bars at top
Also take $d_{req} = 13.5'' \Rightarrow h = 16''$ (assuming one layer of steel)

(ii) Steel ratio will exceed ρ_{max} if designed as singly reinforced beam with b = 12″, $h = 14'' \Rightarrow d = 11.5''$

So the section should be designed as doubly reinforced beam.

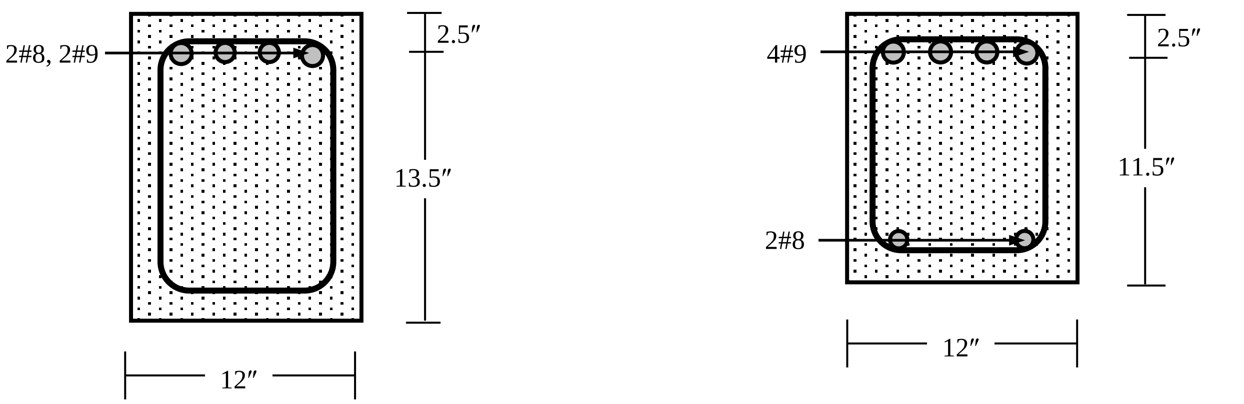
$\therefore$ Ultimate moment, $M_u = (3.1 + 1.4 \times 0.18) \times 10^2/2 = 167.3$ k-ft $\Rightarrow M_n = 167.3/0.9 = 185.8$ k-ft

$M_1 = \rho_s f_y (1 - 0.59\ \rho_s f_y/f_c')\ bd^2 = (1.037) \times 12 \times 11.5^2/12 = 137.2$ k-ft
$\therefore M_2 = M_n - M_1 = 185.8 - 137.2 = 48.6$ k-ft

For M_1, $A_{s1} = 0.0213 \times 12 \times 11.5 = 2.94$ in^2, and $c = A_{s1} f_y/(\alpha f_c' b) = 2.94 \times 60/(0.72 \times 4 \times 12) = 5.10''$

For moment M_2, $A_{s2} = 48.6 \times 12/\{60 \times (11.5 - 2.5)\} = 1.08$ in^2
$\Rightarrow A_s = A_{s1} + A_{s2} = 4.02$ in^2, $\therefore$ Use 4 #9 Bars at top

Since $c = 6.13''$, $f_s' = 87\ (c - d')/c = 44.4$ ksi $< f_y$
$\therefore A_s' = 48.6 \times 12/\{44.4 \times (11.5 - 2.5)\} = 1.46$ in^2, $\therefore$ Use 2 #8 Bars at bottom

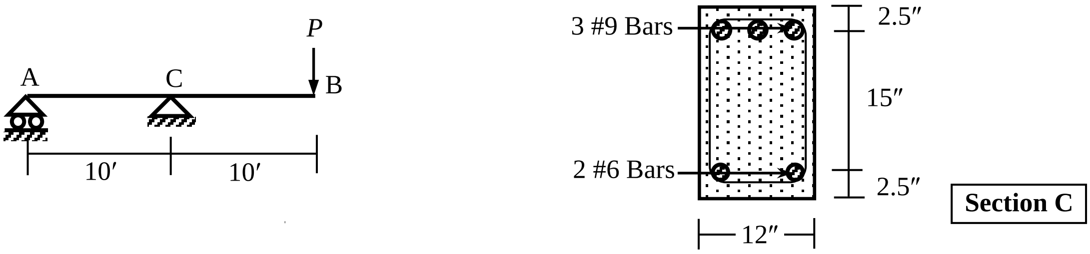


Section B designed as (i) Singly Reinforced, (ii) Doubly Reinforced Section

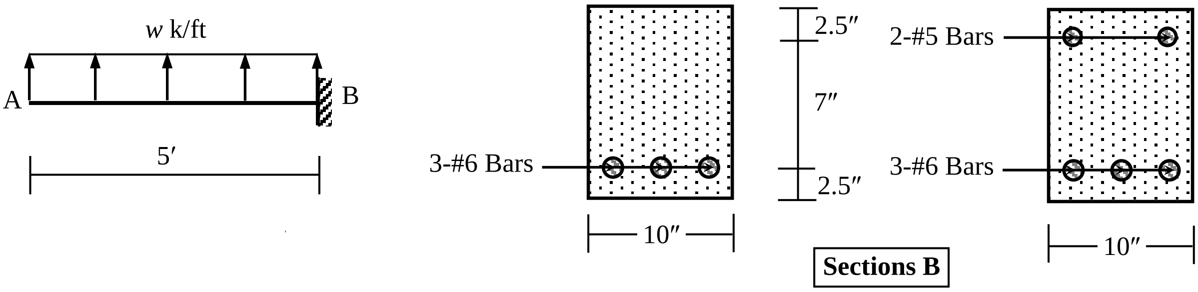
Questions and Problems (3)

1. (i) What is the Ultimate Strength Design (USD) of Reinforced Concrete? Mention its differences from WSD.
   (ii) What are the factors α and β in USD? Explain their variations with the ultimate strength of concrete.
   (iii) What is Whitney's stress block? Explain why it is used in USD.
   (iv) What is the balanced steel ratio (ρ_b)? Why does the ACI recommend a maximum steel ratio less than ρ_b?
   (v) What are the load and resistance factors? Explain why they are used in USD.

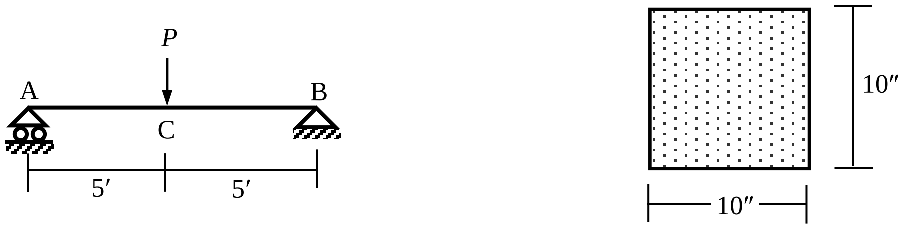
2. Use USD to calculate the maximum allowable <u>live load P</u> for the RC beam ACB shown in the figures below. Include the self-weight of the section also while calculating the bending moment [Given: f_c' = 3 ksi, f_y = 40 ksi].



3. Use the USD to calculate the maximum allowable <u>live load w k/ft</u> on the RC cantilever beam (with sections B) shown below [Given: f_c' = 3 ksi, f_y = 40 ksi].

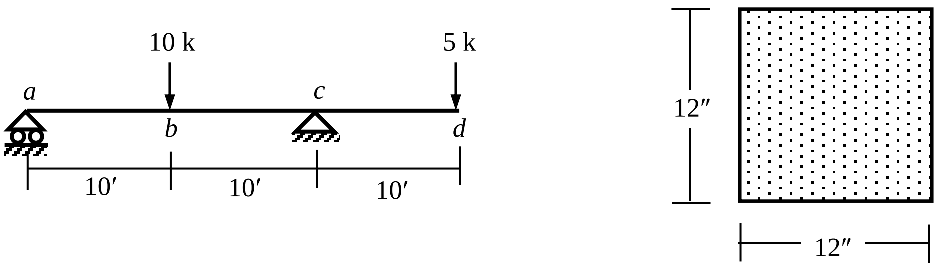


4. Use USD to design section C of the simply supported RC beam ACB shown below if <u>working live loads</u> (i) P = 0, (ii) P = 5 kips, (iii) P = 10 kips. Include self-weight of the beam also [Given: f_c' = 3 ksi, f_y = 40 ksi].



5. Design a 30-ft span simply supported beam loaded (in addition to self-weight) with <u>working dead load w_D = 1 k/ft, and working live load w_L = 1 k/ft</u>, as a
   (i) singly reinforced beam assuming a steel ratio of ρ_{max},
   (ii) doubly reinforced beam assuming a depth 3″ less than the one calculated in (i) [Given: f_c' = 4 ksi, f_y = 60 ksi].

6. Design the beam *abcd* with <u>working live loads shown below</u> (in addition to self weight) for maximum positive and negative moments, if it has a (12″ × 12″) cross-section shown alongside [Given: f_c' = 3 ksi, f_y = 50 ksi].



Following are some of the theoretical questions discussed so far, with guidelines for their answers.

- The examination questions can be different or mixed or parts (based on the same topics/concepts).
- Don't memorize/copy this language, just follow the points and read books to prepare your own answers.

1. What is RC? Explain why steel and concrete are used in conjunction in RC.
   - In Reinforced Concrete (RC), steel is used to strengthen the section against tension in particular (also to resist compression, shear, and control crack, deflection).
   - Concrete protects steel against weather, corrosion, fire, etc.
   - Steel and concrete have very good bond and similar coefficients of thermal expansion.

2. Explain the dependence of stress-strain behavior of concrete and steel on their ultimate strength.
   - Draw the σ-ε graphs of concrete and steel showing variations with ultimate strengths.
   - Increase in f_c' makes concrete stiffer and less ductile.
   - Increase in f_y makes steel less ductile, reduces (even vanishes) the yield region but doesn't affect the modulus of elasticity.

3. What is a 'transformed' RC section? Explain with reference to cracked and uncracked section.
   - In a transformed RC section, reinforcements are 'transformed' to 'equivalent' concrete areas so that the entire section behaves like a plain concrete section rather than a composite.
   - Draw the transformed cracked and uncracked sections and explain the term 'n'.

4. Explain the difference between analysis and design of an RC section.
   - Analysis of an RC section is the calculation of its force/moment/load carrying capacity based on sectional dimensions, steel areas and material properties. It is often referred to as review of a section and is particularly useful in assessing the capacity of an existing structure.
   - Design of an RC section is the calculation of its sectional dimensions and steel areas based on material and structural properties as well as applied loads. It is particularly useful in choosing a new structure/section to withstand given loads.

5. What is a doubly reinforced RC section? Explain how it differs from a singly reinforced section.
   - If the concrete in the compression zone of an RC section is not sufficient to withstand the compressive forces, additional compressive reinforcements may be provided to assist it. Thus, in a doubly reinforced RC section, steel is provided in the tension as well as compression zones of concrete.
   - Stresses in the tension and compression reinforcements of a doubly reinforced beam are different. In USD, the compression reinforcement may or may not yield.
   - Doubly reinforced sections are more common because in addition to resisting compression, the compressive steel is required to control crack and deflection as well as to bind stirrups.

6. Why does the ACI recommend that in WSD, the value of compressive stress in steel (f_s') be taken as twice the value calculated from elastic analysis?
   - Derive the expression for f_s' using stress and strain distributions over the section.
   - Material nonlinearity in concrete lowers the stress in it, inducing more stress in the compression reinforcement [Draw the σ-ε graphs of concrete and steel to explain].
   - Creep (long-term deflection) causes large strains in concrete, resulting in larger stresses in the compression steel due to the resistance it provides to such strains.

7. Show the variations of stress and strain over an RC section as it is stressed gradually from uncracked to cracked and ultimate failure condition.
   - Draw the appropriate diagrams for variation of strain and stress for uncracked, cracked (WSD) and ultimate failure (USD) conditions.
   - For small moments (and stresses), the tensile stresses are within the tensile strength of concrete and the section remains uncracked.
   - For larger moments (and stresses), the tensile stresses exceed the tensile strength of concrete but the compressive stresses in concrete and tensile/compressive stresses in steel remain within elastic limits.
   - For even larger moments (and stresses), the tensile stress in steel reaches its yield point and compressive stress in concrete exceeds the proportional limit so that the σ-ε relationships are no longer linear.

8. What is the USD method of RC design? Mention its differences from the WSD method.
   - In Ultimate Strength Design (USD) method, RC sections are designed to survive the maximum expected loads within their lifetime without exceeding material strengths (i.e., suffering failure), but exceeding elastic limits.
   - In Working Stress Design (WSD) method, RC sections are designed to withstand the maximum working loads without exceeding some predefined 'safe' stress levels (incorporating some 'safety factors'). Usually these stresses are within the elastic limits of the materials.
   - The USD is a more rational and usually more economical method than the WSD.

9. What are the factors α and β used in USD? Explain their variations with the ultimate strength of concrete.
    - The α and β factors are used in the USD method account for the material nonlinearity in calculating the compressive forces in concrete and the moments induced
    - Show with figures for concrete stress distribution over compression zone.
    - Draw the variations of α and β with f_c'.
    - Draw the σ-ε diagrams of concrete of various strengths. Concretes of higher strength tend to be stiffer and less ductile.
    - Since α is the ratio of the average stress to maximum, this tends to decrease with f_c'.
    - Since β is a factor indicating the distance of the resultant force from ultimate strain, this also tends to decrease with f_c'.

10. What is Whitney's stress block? Explain why it is used in USD.
    - The nonlinear σ-ε relationship for concrete is difficult to model, requiring different factors (like α and β introduced in USD) to compute the compressive forces and moments induced.
    - As an alternative, Whitney proposed a simple stress block whose formulation and application is much easier.
    - Derive the appropriate parameters for Whitney's stress block (e.g., '*a*' and 0.85).

11. What are the maximum and minimum allowable steel ratios used in RC? Explain why they are used.
    - Write the formulae for p_b, p_{max}, p_{min} and explain the terms in them.
    - If the steel ratio exceeds p_b, the failure of the RC section will be initiated by the crushing of concrete rather than the yielding of steel. This is highly undesirable due to the sudden and explosive nature of concrete crushing compared to the more ductile mode of steel yielding [Explain them using σ-ε diagrams of steel and concrete].
    - If the steel ratio is set below p_{min}, the tension steel will not be sufficient to resist the tensile forces induced and the structural performance of the RC section will not be better than the behavior of a pure concrete section. Therefore, failure of the section will follow immediately after the tension cracking of the section.

12. What are the load and resistance factors? Explain why they are used in USD.
    - Load factors are provided to account for the uncertainty in assumed working loads, increasing them to the maximum possible loads (statistically expected) during the lifetime of the structure. In the USD, RC sections are designed to survive these maximum loads without suffering structural failure [Give examples for DL, LL, Wind, EQ etc].
    - Resistance factors are provided to consider the uncertainty in the material strengths and structural dimensions, reducing them to the statistically expected minimum levels depending on the construction of the materials and importance of the structures [Give examples for moment, shear and axial forces for both steel and concrete].

Analysis and Design of T- and L-Beam

Reinforced Concrete floors, roofs, decks, etc. are almost always monolithic. Forms are built for beam soffits (lower parts), sides and underside of slabs, the entire construction being poured at once from the bottom of the deepest beam to the top of the slab. It is therefore evident that a part of the slab will act with the upper part of the beam to resist longitudinal compression. The resulting beam cross-section is T-(or L-) shaped rather than rectangular; where slab forms the beam flange, and the part of the beam projecting below the slab forms the web or stem.

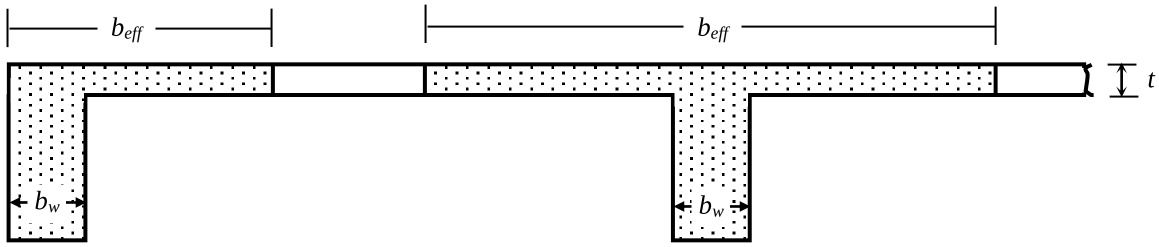


Fig. 7.1: Effective Widths of L- and T-Beams

The following cause the stresses in the slab to differ from the concrete stress and strength in rectangular beams.

* In addition to longitudinal stresses, the upper part of the beam is stressed laterally due to slab action. This may cause the compressive strength of slab's concrete to increase (due to compression) or decrease (due to tension). This effect is usually not taken into consideration in design.

* The flange is not uniformly stressed across its width (due to shear deformation of the flange); i.e., elements of the flange midway between the beam stems are stressed less than the elements directly over the stem. It is convenient in design to use an 'effective flange width' (b_{eff}) which may be smaller than the actual flange width, but is considered to be uniformly stressed at the maximum value. This 'effective width' has been found to depend primarily on the beam span and relative thickness of the slab. The ACI recommends the following values of the effective width for the T- and L-beams

T-beam b_{eff} is $\geq L/4$, $(16t + b_w)$, and (c/c distance between adjacent beams) …………………..(7.1)

L-beam b_{eff} is $\geq (L/12 + b_w)$, $(6t + b_w)$, and (b_w + half the clear distance between adjacent beams) …………….(7.2)

where L = Beam span between consecutive points of inflection (i.e., span where bending moment is positive),
t = Slab thickness, and b_w = Width of web or stem

The neutral axis of the beam may be either in the flange or the web, depending proportions of the cross-section, the amount of tensile steel, and material strength. If it is within the slab, the beam can be analyzed as a rectangular beam of width b_{eff}, but a T-beam analysis is necessary otherwise.

Working Stress Design of T- (and L-) Beam

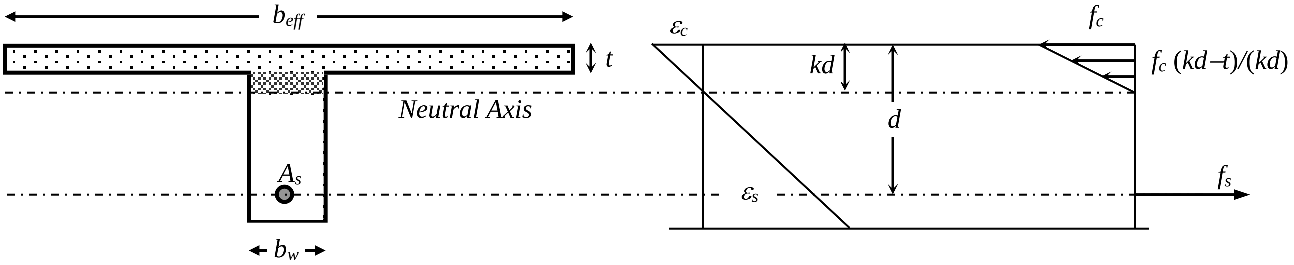


Fig. 7.2: Working Strain and Stress distribution in T-Beam

The equation of k [$= n/(n+r)$] for rectangular beam can be used in T-beam also, provided the value of n ($= E_s/E_c$) and r ($= f_s/f_c$) are known. However, the actual r is usually not known for T-beams, although the maximum allowable stresses are, since the compressive area provided by the slab is so large that the actual f_c is some fraction of its allowable value.

In Fig. 7.2, neglecting the small (darkened) compressive area in the web, the total compressive force in concrete is

$$C \cong \{(f_c + f_c\,(kd-t)/(kd))/2\}(bt) = f_c\,\{1-t/(2kd)\}(b_{eff}\,t) \qquad \text{.................(7.3)}$$

which is equal to the tensile force T ($= A_s f_s$) in steel. ....(7.4)

Obtaining (f_s/f_c) from the combination of Eqs. (7.3) and (7.4), the following expression for k is obtained,

$$k = \{n\rho_s + (t/d)^2/2\}/\{n\rho_s + (t/d)\} \qquad \text{.................(7.5)}$$

The distance to the resultant of compression from the upper face of the beam is

$$z = (3kd - 2t)/(2kd - t)\,(t/3) \qquad \text{.................(7.6)}$$

and the moment arm of the couple (formed by C and T) is, $jd = d - z$ ....(7.7)

The resisting moments of steel and concrete are given by

$$M_s = T\,jd = A_s f_s\,jd \qquad \text{................. (7.8)}$$

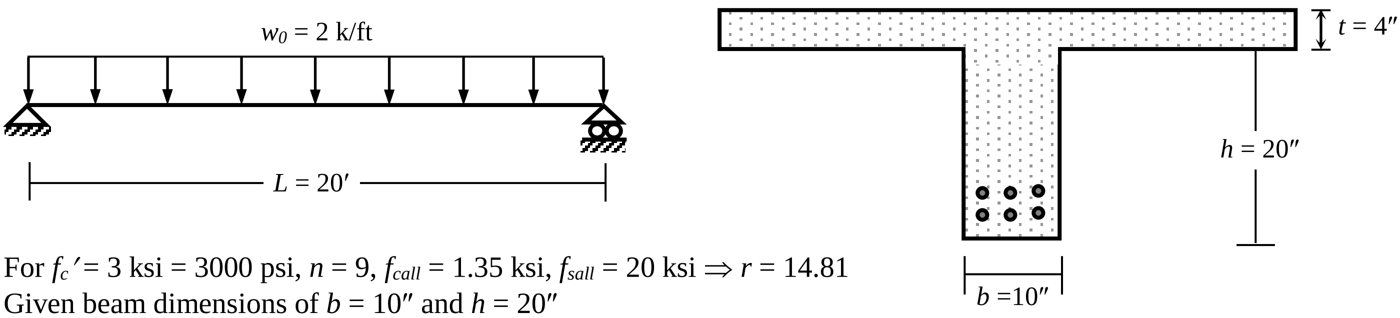
$$M_c = C\,jd = f_c\,\{1-t/(2kd)\}(b_{eff}\,t)\,jd \qquad \text{.................(7.9)}$$

Since the Eq. (7.5) for k requires the steel ratio ρ_s ($= A_s/b_{eff}\,d$), simplified approximate equation for A_s can be derived for design problems based on assumed $z \cong t/2$, which is always a conservative estimate; i.e.,

$$A_s \cong M_s/\{f_s(d - t/2)\} \qquad \text{.................(7.10)}$$

Example 7.1

Use the WSD Method to design the simply supported T-beam loaded as shown below, in addition to its self-weight if it is part of a beam system carrying a 4″ thick slab (with FF = 30 psf, RW = 80 psf and LL = 40 psf) a transverse distance 10′ c/c apart [Given: $f_c{'}$ = 3 ksi, f_{call} = 1.35 ksi, f_{sall} = 20 ksi].



For $f_c{'}$ = 3 ksi = 3000 psi, n = 9, f_{call} = 1.35 ksi, f_{sall} = 20 ksi $\Rightarrow r$ = 14.81
Given beam dimensions of b = 10″ and h = 20″

Weight from slab = 4/12 × 150 + 30 + 80 + 40 = 200 psf = 0.20 ksf
For beam c/c distance = 10′, the superimposed load per ft is w_0 = 0.20 × 10 = 2.0 k/ft
Beam self-weight = $(10 \times 20/12^2) \times 0.15$ = 0.208 k/ft $\Rightarrow$ Total load w = 2 + 0.208 = 2.208 k/ft
$\Rightarrow M_{max} = wL^2/8 = 2.208 \times 20^2/8$ = 110.4 k-ft

Also b_{eff} = Minimum of [L/4 (= 60″), $16t + b_w$ (= 64 + 10 = 74″), and c/c distance (= 120″)] = 60″
$\therefore$ Assuming two layers of bottom steel, d = 20 + 4 – 4 = 20″, $A_s \cong 110.4 \times 12/\{20\,(20 - 4/2)\} = 3.68$ in^2

$\therefore n\rho_s = 9 \times 3.68/(60 \times 20) = 0.0276 \Rightarrow k = 0.209$, $kd = 0.209 \times 20 = 4.18″ > t$
$\therefore z = (3 \times 4.18 - 2 \times 4)/(2 \times 4.18 - 4) \times (4/3) = 1.39″ \Rightarrow jd = d - z = 18.61″$

$\therefore A_s = 110.4 \times 12/(20 \times 18.61) = 3.56$ in^2; i.e., Use 6 #7 Bars in two layers
and concrete stress $f_c = A_s f_s/[\{1-t/(2kd)\}(b_{eff}\,t)] = 3.56 \times 20/(1-4/8.36)(60 \times 4) = 0.57$ ksi $< f_{call}$ (OK)

Ultimate Strength Design of T- (and L-) Beams

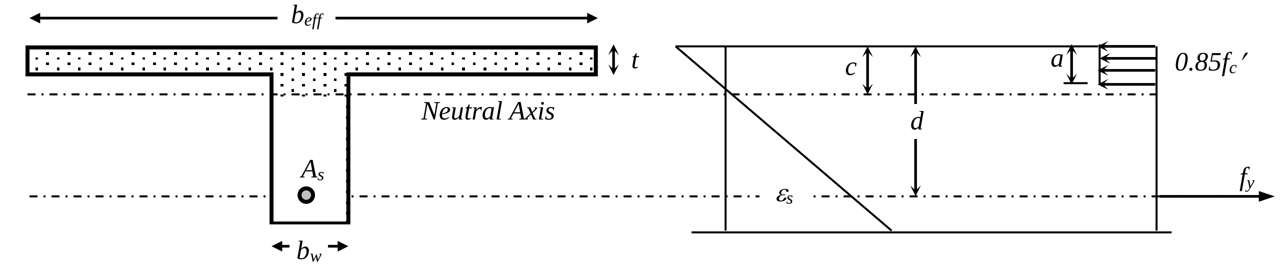


Fig. 7.3: Ultimate Strain and Stress distribution in T-Beam

In USD, the effective width b_{eff} of T-beam is governed by the same conditions as described in WSD. It is convenient to divide the total tensile steel into two parts. The first part, A_{sf}, represents the steel area required to balance the compressive force in the overhanging portions of the flange that are stressed to $0.85f_c'$. Thus,

$A_{sf}\, f_y = 0.85f_c'(b_{eff} - b_w)t \Rightarrow A_{sf} = 0.85f_c'(b_{eff} - b_w)t \,/\, f_y$ ……………….(7.11)

The forces $A_{sf}\, f_y$ and $0.85f_c'(b_{eff} - b_w)t$ act with a moment arm $(d - t/2)$ to provide the nominal resisting moment

$M_{nf} = A_{sf}\, f_y\,(d - t/2)$ ……………….(7.12)

Tension in the remaining steel area ($A_{sw} = A_s - A_{sf}$) is balanced by the compression in the rectangular portion of the beam-web. The depth of the equivalent rectangular stress block is obtained from

$A_{sw}\, f_y = 0.85f_c' a b_w \Rightarrow a = A_{sw}\, f_y /(0.85f_c' b_w)$ ……………….(7.13)

An additional moment M_{nw} is thus provided by forces $A_{sw}\, f_y$ and $0.85f_c' a b_w$ acting with the moment arm $(d - a/2)$; i.e.,

$M_{nw} = A_{sw}\, f_y\,(d - a/2)$ ……………….(7.14)

Therefore, total nominal moment capacity is $M_n = M_{nf} + M_{nw} = A_{sf}\, f_y\,(d - t/2) + A_{sw}\, f_y\,(d - a/2)$ ……………….(7.15)

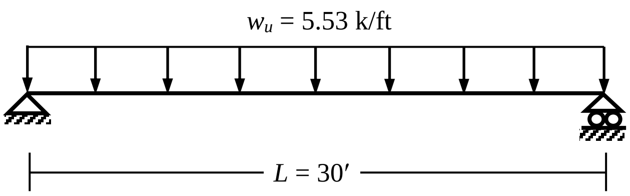
To ensure ductile failure (i.e., steel yielding prior to concrete crushing) of the beam, ratio of web steel (that acts with concrete block of depth c) has to be less than the balanced steel ratio; i.e., $\rho_{sw} \le \rho_b \Rightarrow \rho_s \le \rho_b + \rho_{sf}$

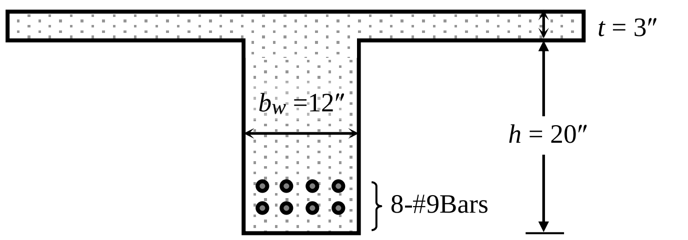
According to ACI, $\rho_s \le 0.75(\rho_b + \rho_{sf})$, and $\ge \rho_{min}$ ……………….(7.16)

where all steel ratios are expressed in terms of the rectangular (web) portion of the beam [$\rho_s = A_s/b_w d$, $\rho_{sf} = A_{sf}/b_w d$].

<u>Example 7.2</u>

Use USD to design the simply supported T-beam loaded as shown below (in addition to its self-weight) if it is part of a beam system carrying a 3″ thick slab (with FF = 30 psf, RW = 120 psf, LL = 150 psf) a transverse distance 10′ c/c apart [Given: f_c' = 3 ksi, f_y = 60 ksi].





Given f_c' = 3 ksi, f_y = 60 ksi, and beam dimensions of b_w = 12″ and h = 20″

Factored weight from slab = 1.4 × (3/12 × 150 + 30 + 120) + 1.7 × 150 = 517.5 psf = 0.518 ksf
For beam c/c distance = 10′, the superimposed load per ft is w_0 = 0.518 × 10 = 5.18 k/ft
Beam self-weight = (12 × 20/12^2) × 0.15 = 0.25 k/ft ⇒ Total load w_u = 5.18 + 1.4 × 0.25 = 5.53 k/ft
∴ Ultimate moment, $M_u = w_u L^2/8$ = 5.53 × 30^2/8 = 621.6 k-ft ⇒ Required Nominal moment, $M_n = M_u/\phi$ = 690.6 k-ft

b_{eff} = Minimum of [L/4 (= 360/4 = 90″), 16t + b_w (= 48 + 12 = 60″), c/c distance (= 10 × 12 = 120″)] ⇒ b_{eff} = 60″
∴ Assuming two layers of bottom steel ⇒ d = 20 + 3 – 4 = 19″
$a = d\,[1 - \sqrt{\{1 - 2M_n/(0.85f_c'\, b_{eff}\, d^2)\}}]$ = 19 [1– √{1 – 2 × 690.6 × 12 /(0.85 × 3 × 60 × 19^2)}] = 3.10″ > t
∴ Section acts like a T-Beam, with $A_{sf} = 0.85f_c'(b_{eff} - b_w)t/f_y$ = 2.55 × (60 – 12) × 3/60 = 6.12 in^2
$M_{nf} = A_{sf}\, f_y\,(d - t/2)$ = 6.12 × 60 × (19 – 3/2)/12 = 535.5 k-ft

∴ $M_{fw} = M_n - M_{nf}$ = 690.6 – 535.5 = 155.1 k-ft
a = 19 [1–√{1–2 × 155.1 × 12 /(0.85 × 3 × 12 × 19^2)}] = 3.53″, $A_{sw} = 0.85f_c' a b_w/f_y$ = 0.85 × 3 × 3.53 × 12/60 = 1.80 in^2
∴ $A_s = A_{sf} + A_{sw}$ = 6.12 + 1.80 = 7.92 in^2; i.e., <u>Use 8 #9 Bars in two layers</u>
⇒ $\rho_{sw} = A_s/(b_w d)$ = 8.00/(12 × 19) = 0.0351, while $\rho_{sw(max)}$ = 0.75 {0.0213 + 6.12/(12 × 19)} = 0.0361
[Note: Approximately assuming $a = t \Rightarrow A_s \cong M_n/[f_y\,(d - t/2)]$ = 690.6 × 12/[60 (19 – 3/2)] = 7.89 in^2]

Design of Inverted Beams

Deep beams spanning over large lengths under slabs across a room definitely do not present the best aesthetic design from architectural point of view. Inverted beams; i.e., beams above slabs, are sometimes provided in such cases to present better aesthetic views from inside the room. Although such beams may not be suitable for floors to be inhabited, they can be used on roof slabs as well as to improve audience views, for example in large auditoriums.

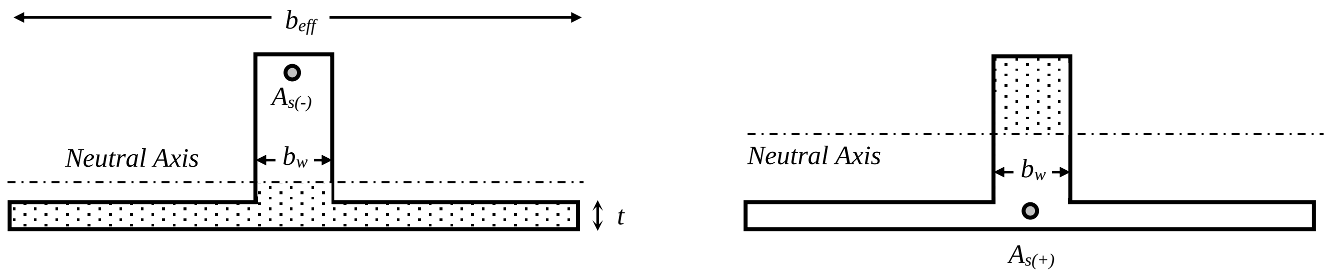


Fig. 7.4: Inverted beam subjected to (a) Negative Moment, (b) Positive Moment

As shown in Fig. 7.4, the behavior of inverted beams is similar to T-beams, only differing in terms of the moment they are subjected to. While they behave as inverted T-beams (of width b_{eff}) when subjected to negative moment [Fig. 7.4(a)], their behavior under positive moment is the same as rectangular beam of width b_w [Fig. 7.4(b)]. One structural advantage of such beams is that the large concrete area provided by slab is active in resisting compressive stresses due to negative moment [Fig. 7.4(a)], which is often larger than the positive moments in typical continuous beams.

Design Moments of Continuous Beams

<u>Load Combination using Influence Lines</u>

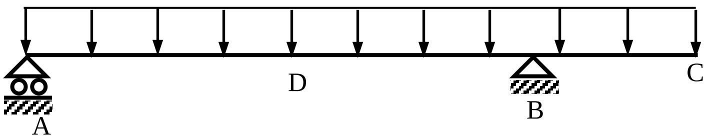


Fig. 7.5(a): Loading for maximum M_B

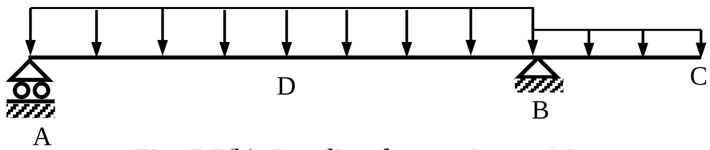


Fig. 7.5(b): Loading for maximum M_D

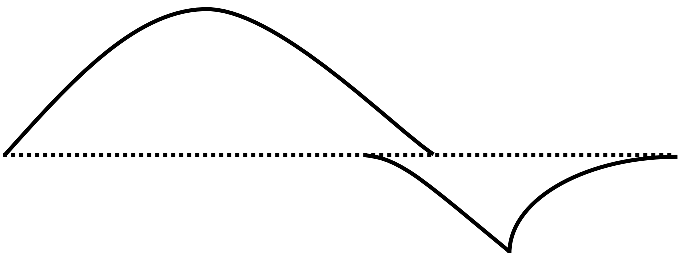


Fig. 7.5(c): Design Moment Diagram

<u>Design Moments using ACI Coefficients</u>

For maximum allowable LL/DL = 3, and maximum allowable adjacent span difference = 20%

Using L = clear span for $M_{(+)}$ and average of two adjacent clear spans for $M_{(-)}$

1. Positive Moments
   (i) For End Spans
      (a) If discontinuous end is unrestrained, $M_{(+)} = wL^2/11$
      (b) If discontinuous end is restrained, $M_{(+)} = wL^2/14$
   (ii) For Interior Spans, $M_{(+)} = wL^2/16$

2. Negative Moments
   (i) At the exterior face of first interior supports
      (a) Two spans, $M_{(-)} = wL^2/9$
      (b) More than two spans, $M_{(-)} = wL^2/10$
   (ii) At the other faces of interior supports, $M_{(-)} = wL^2/11$
   (iii) For spans not exceeding 10′, of where columns are much stiffer than beams, $M_{(-)} = wL^2/12$
   (iv) At the interior faces of exterior supports
      (a) If the support is a beam, $M_{(-)} = wL^2/24$
      (b) If the support is a column, $M_{(-)} = wL^2/16$

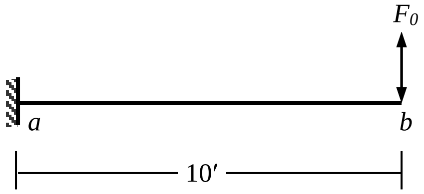
Questions and Problems (4)

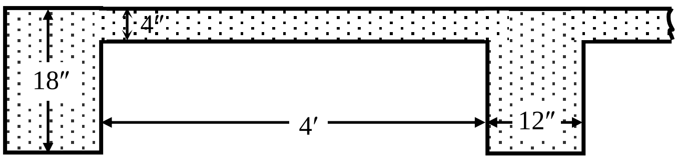
1. (i) Mention the differences and advantages of designing beams as T-beams compared to rectangular beams.
   (ii) Explain the differences between flexural stress distribution over T- and rectangular beams (and their effects).
   (iii) Mention and explain the ACI-recommended effective widths of T- and L-Beams.

* Explain the differences between flexural stress distribution over T- and rectangular beams (and their effects).
- Two-dimensional stress distribution due to tension/compression from slab (effect on f_c').
- Variation of stresses across the width of beam due to shear deformations (effect on b_{eff}).

* Mention and explain the ACI-recommended effective widths of T- and L-Beams.
- Variation of stress across the width of the beams.
- Mention the T-beam widths and L-beam widths.

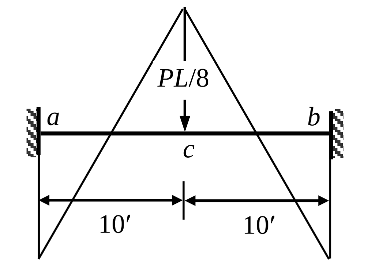
2. Use the WSD method to design the end beam (L-section) at '*a*' of the cantilever beam *ab* subjected to F_0 = 10 k acting (i) downward, (ii) upward on the beam (in addition to slab self weight plus FF = 20 psf, RW = 50 psf, LL = 30 psf and beam self weight) [Given: f_c' = 3 ksi, f_s = 20 ksi].

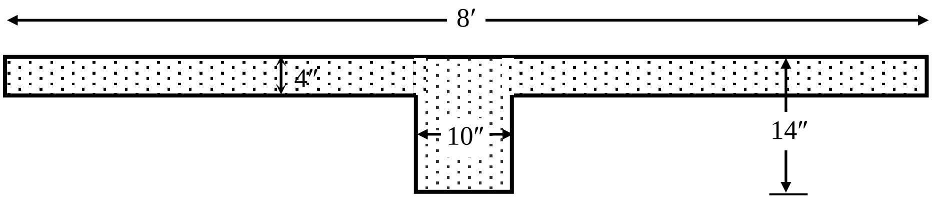




Beam and Slab Sections

3. Use the WSD method to design the section *c* of the fixed-ended beam *ab* shown in the figures below if (i) P = 5 kips, (ii) P = 10 kips. Exclude the weights from the slab and beam [Given: f_c' = 3 ksi, f_s = 18 ksi].





Beam and Slab (c/c) Section

4. Answer the Question 2, using USD, assuming F_0 to be live load [Given: f_c' = 3 ksi, f_y = 50 ksi].

5. Answer the Question 3, using USD, assuming P to be live load [Given: f_c' = 3 ksi, f_y = 40 ksi].

Shear Force and Stress in Plain Concrete

Other than axial force, bending moment and the resulting normal (i.e., axial and flexural) stresses, beams must also have an adequate safety margin against other types of failure, some of which may be more dangerous than flexural failure. Shear failure of RC, more properly called *Diagonal Tension Failure* (due to the mode of failure, i.e., as the principal tensile stress that acts diagonally, as shown in Figs. 8.1 and 8.2) is one such example. In contrast with the nature of flexural failure, it may occur suddenly (without any warning of distress). Therefore, RC beams are generally provided with *shear reinforcement* to ensure flexural failure occurs before shear failure.

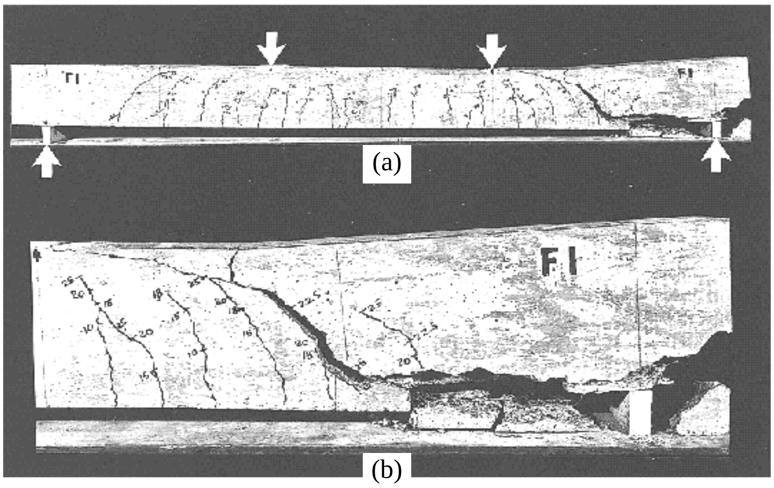


Fig. 8.1: Shear Failure of concrete beam: (a) Overall View, (b) Detail near right support (Nilson)

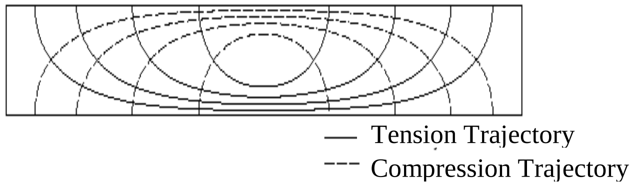
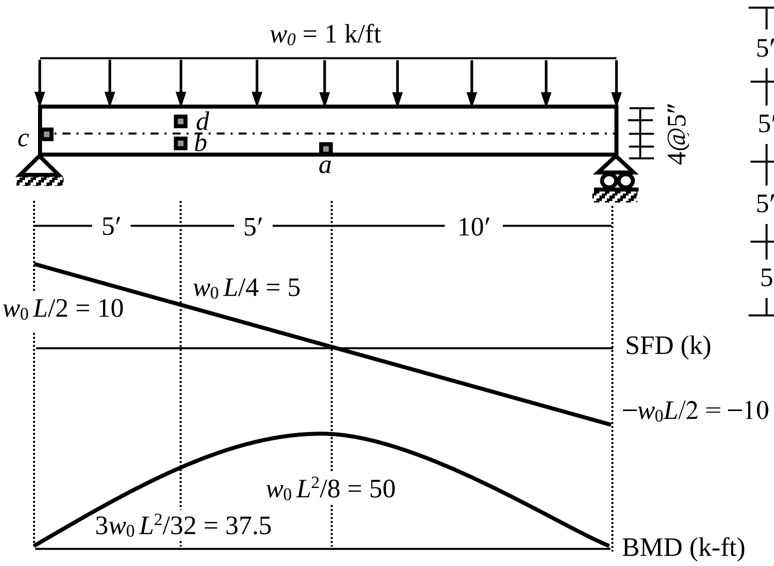
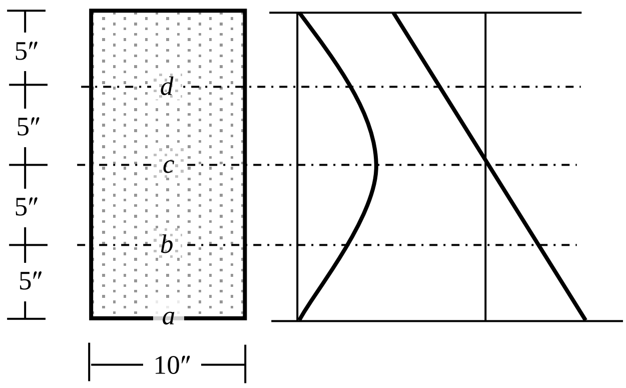


Fig. 8.2: Stress Trajectories of Diagonal Tension and Compression (Nilson)

Example 8.1

Calculate the magnitude and direction of the maximum tensile stresses at points *a*, *b*, *c*, *d* of the simply supported RC beam loaded as shown below.

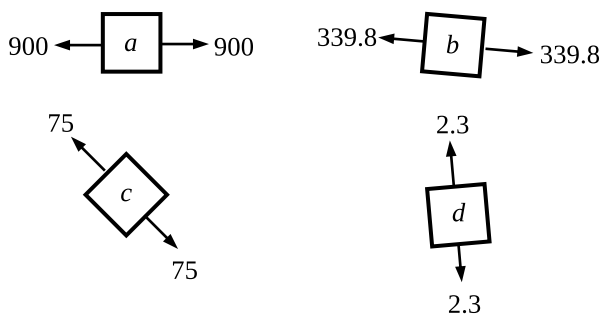




$\sigma_1 = \sigma/2 + \sqrt{(\sigma/2)^2 + (\tau)^2}$

$\tan 2\theta = 2\tau/\sigma$

Point	V (k)	τ (psi)	M (k-ft)	σ (psi)	σ_1 (psi)	θ
a	0	0.00	50	900.0	900.0	0°
b	5	28.13	37.5	337.5	339.8	4.7°
c	10	75.00	0	0.0	75.0	45°
d	5	28.13	37.5	–337.5	2.3	85.3°



Shear Force and Stress in Reinforced Concrete

<u>RC Sections without Shear Reinforcement</u>

The behavior of plain concrete sections is similar to homogeneous sections. A tension crack forms where the tensile stresses are largest (usually a *Flexural crack*, where flexural stresses are largest) and quickly leads to the collapse of the beam. But the behavior of Reinforced Concrete sections is quite different because longitudinal reinforcements are provided to take care of these tensile stresses. However, these reinforcements are not designed to counter the diagonal tensile stresses that occur elsewhere, resulting in the formation of diagonal cracks, which may eventually lead to the beam to fail. If flexural stresses are negligibly small at the particular location, the diagonal tensile stresses [Fig. 8.3(a)] are inclined at about 45° and are numerically equal to the shear stresses (τ), with a maximum at the neutral axis. Diagonal tensile cracks, called the *Web-Shear cracks*, form as a result at or near the neutral axis, approximately at an 'average shear stress' (shear force divided by the effective beam area *bd*) of

$v_{crw} = 3.5\sqrt{f_c'}$ (in psi) ……………………………………………….(8.1)

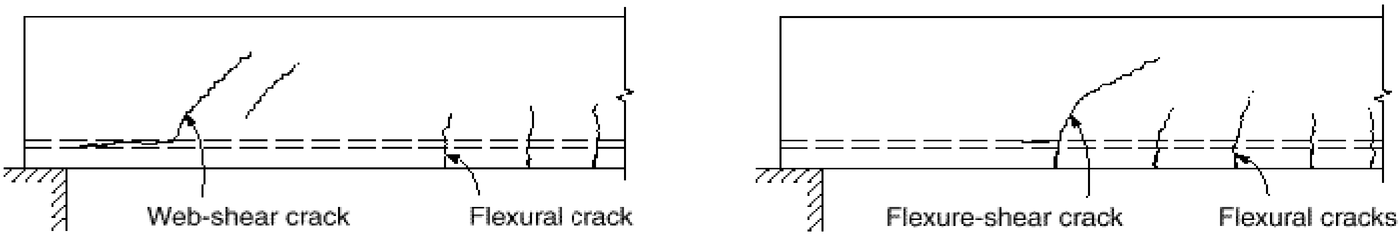


Fig. 8.3: Diagonal Cracking in RC Beams (a) Web-shear cracking, (b) Flexural-shear cracking (Nilson)

However, the situation is different when both shear force and bending moment have large values. In such cases, usually the flexural tension-cracks form first, and are controlled by flexural reinforcement. However, they tend to grow in length and width when their values exceed the tensile strength of concrete. These cracks, called the *Flexural-Shear cracks* [Fig. 8.3(b)], are more common than *Web-Shear cracks*, and form at a lower 'shear stress', due to the pre-existing tension crack of the concrete section and increased diagonal stress due to flexural stresses

$v_{crf} = 1.9\sqrt{f_c'}$ (in psi) ……………………………………………….(8.2)

Eqs. (8.1) and (8.2) indicate that the stress where diagonal cracks develop depends on the ratio of the shear stress (τ) and flexural stress (σ), which is proportional to the non-dimensional term Vd/M, and has been empirically found to be

$v_{cr} = 1.9\sqrt{f_c'} + 2500\rho_s\,(Vd/M) \leq 3.5\sqrt{f_c'}$ (in psi) ……………………………………………….(8.3)

where $\rho_s = (A_s/bd)$ is the longitudinal steel ratio at the section. The presence of longitudinal rod increases the shear resistance of the section, evidently because they lead to smaller and narrower flexural cracks, leaving a larger uncracked concrete area available to resist shear. Conservatively, however, this effect is ignored in design and the value of v_{cr} is taken equal to

$v_{cr} = 2\sqrt{f_c'}$ (in USD) and $1.1\sqrt{f_c'}$ (in WSD) ……………………………………………….(8.4)

Fig. 8.4 shows components of the vertical forces that combine to provide the resistance to the external shear forces on a RC section. These include V_{cz} = Vertical component of the uncracked portion of concrete, V_{iy} (= Vertical component of the interlock force V_i) and V_d (= Bearing force between concrete and longitudinal steel, acting as a dowel), i.e., the external force V_{ext} is equal to the summation of these components

$V_{ext} = V_{cz} + V_{iy} + V_d$ ……………………………………………….(8.5)

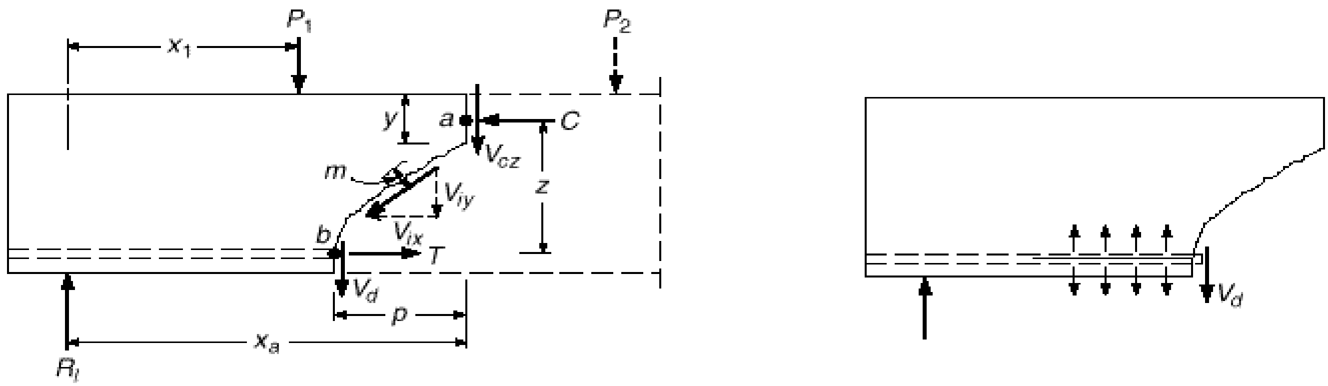


Fig. 8.4: Forces at a diagonal crack in a beam without web reinforcement (Nilson)

RC Sections with Shear Reinforcement

It is often desirable to allow RC members to fully develop its flexural capacity, so that it does not fail prematurely and suddenly due to shear. Therefore, if a fairly large factor of safety relative to available shear strength does not exist, special shear reinforcement, known as *Web Reinforcement*, is used to increase its shear strength. Fig. 8.5 shows various types of web reinforcements used in RC beams. Whereas they may be *vertical stirrups* [Fig. 8.5(a)] or provided by *bent-up longitudinal bars* [Fig. 8.5(b)], they are available in various forms shown in Fig. 8.5(c) (usually they are two- or multiple-legged #3 to #5 bars).

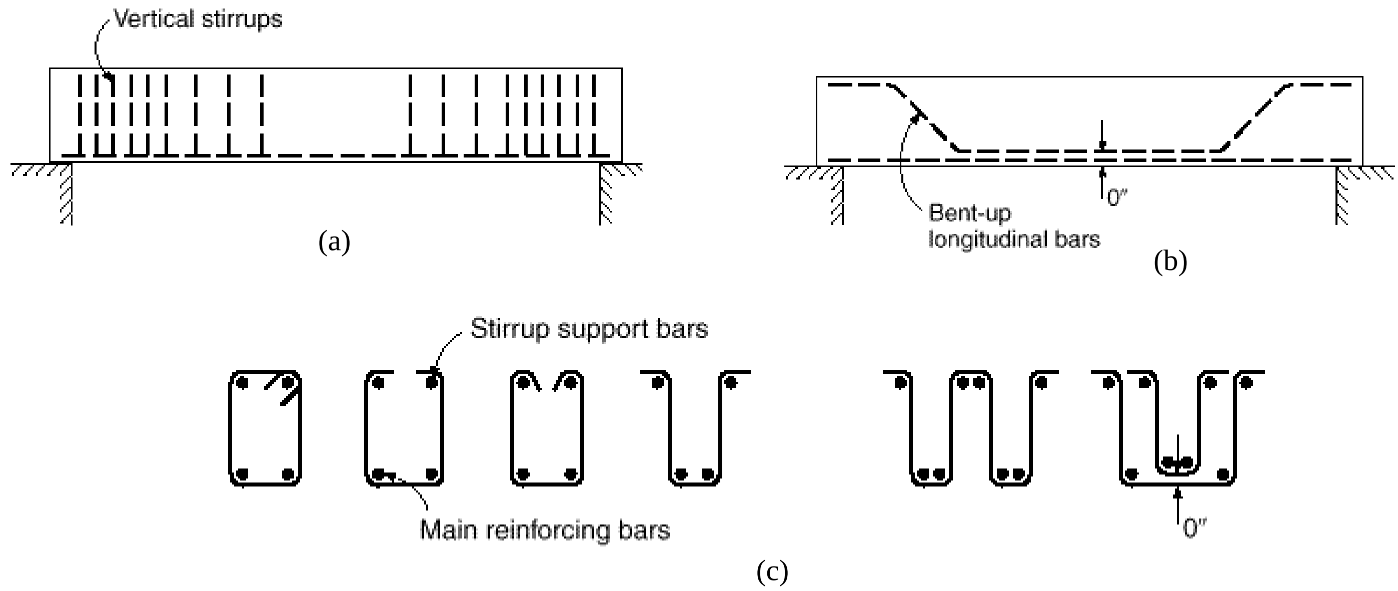


Fig. 8.5: (a) Vertical stirrups, (b) Bent-up longitudinal bars, (c) Various forms of stirrups (Nilson)

Beams with Vertical Stirrups

Web reinforcements have almost no effect prior to the formation of diagonal cracks, and as such the shear stress (or force) to be resisted by concrete remains essentially the same as before the formation of cracks, as given by Eqs. (8.1)~(8.4). However, they enhance the shear resistance of the beam after the formation of diagonal cracks, by (i) restricting lengthening of diagonal cracks into concrete compression zone (thereby increasing V_{cz}), (ii) restricting widening of cracks (thereby increasing the interface force V_{iy}), (iii) tying the longitudinal bars (thereby resisting their splitting and increasing dowel action force V_d), (iv) most importantly, by resisting the external shear force by an additional force V_s, provided by the tensile force in the stirrups.

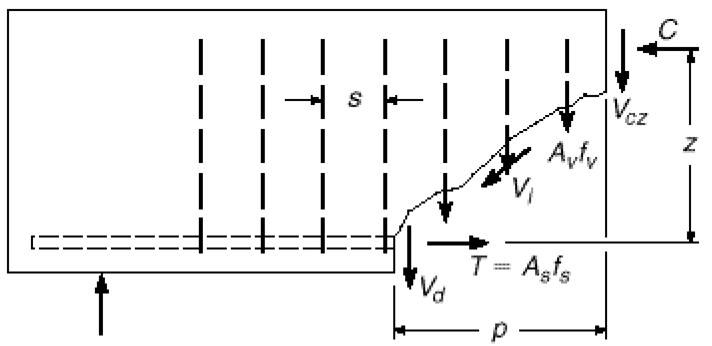


Fig. 8.6: Forces at diagonal crack with Vertical Stirrups (Nilson)

Force per stirrup = $A_v f_v$
Width of crack = p, approximately equal to effective beam depth d.

Spacing of stirrups = S
$\therefore$ Number of stirrups within the crack, $n = d/S$
$\therefore$ Total force from stirrups within crack
$V_s = A_v f_v (d/S)$ ………………....(8.6)

$\therefore V_{ext} = V_{cz} + V_{iy} + V_d + V_s$ ……………....(8.7)

Taking $V_{cz} + V_{iy} + V_d = V_{cr}$
$\Rightarrow V_{ext} = V_{cr} + A_v f_v (d/S)$ ………………...(8.8)

$\Rightarrow S = A_v f_v d/(V_{ext} - V_{cr})$ ………………....(8.9)

Eq. (8.9) gives the longitudinal spacing of vertical stirrups required to resist the applied shear force V_{ext}.

Shear Design Concepts by WSD and USD

Design Location for Shear Force Calculation

For typical support conditions, sections located less than a distance d from the face of the support may be designed for the same shear as that computed at a distance d. However, if concentrated loads act within that distance or if the support reaction causes tension, the critical design section should be at the face of the support.

Stirrup Spacing in terms of Shear Stress

The required spacing S between vertical stirrups [given by Eq. (8.9)] can be rearranged using shear stresses ($v = V/bd$) and strengths (e.g., $v_{cr} = V_{cr}/bd$) instead of using shear forces, according to the following equations

$$S = A_v f_v / \{(v_{ext} - v_c)\, b\} \qquad \text{.....................(8.10)}$$

For bars inclined at an angle α with the horizontal, Eqs. (8.9) and (8.10) are replaced by

$$S = A_v f_v\, d\, (Sin\ \alpha + Cos\ \alpha)/(V_{ext} - V_c) = A_v f_v (Sin\ \alpha + Cos\ \alpha)/\{(v_{ext} - v_c)\, b\} \qquad \text{.....................(8.11)}$$

Minimum Web Reinforcement

If the shear force $V_{ext} \leq V_c$, then theoretically no web reinforcement is required. Even in such a case, the ACI Code requires a provision of at least a minimum area of web reinforcement equal to

$$A_{v,min} = 50\, b_w S/f_y \qquad (f_y \text{ to be taken in psi}) \qquad \text{.....................(8.12)}$$

This is equivalent to designing assuming the stirrups to be designed for a minimum section shear stress of 50 psi.

The provision holds unless $V_{ext} \leq V_c/2$. Exceptions are made for slabs, footings, concrete joist floor, and beams with $h \leq (10'', 2.5t, b_w/2)$, because of their capacity to redistribute internal forces before diagonal tension failure.

Where web reinforcement is needed, the Code requires it to be spaced so that every 45° line, representing a potential diagonal crack extending from the mid-depth $d/2$ to longitudinal bars (shown in Fig. 8.7 for diagonal bars also), is crossed by at least one line of web reinforcement; in addition, the code specifies a maximum stirrup spacing of 24″.

These maximum spacings are halved when $V_s \geq 4\sqrt{f_c'}b_wd$, or $V_n \geq 6\sqrt{f_c'}b_wd$ ($V_w \geq 3\sqrt{f_c'}b_wd$ in WSD).

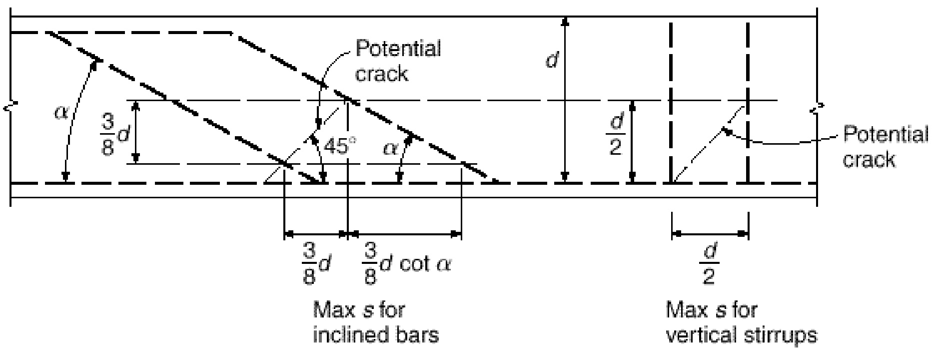


Fig. 8.7: Maximum spacing of web reinforcement as governed by at diagonal crack interception (Nilson)

It is undesirable to space vertical stirrups closer to 4″; the size of the stirrups should be chosen to avoid closer spacing. When vertical stirrups are required over a small distance (e.g., common residential buildings), it is better to space them uniformly. Over a long distance, however (e.g., bridges), it is economical to compute them at several locations.

Maximum Allowable Shear V_s

To avoid excessive crack in beam webs, the ACI Code limits the yield strength of reinforcement to $f_y \leq 60$ ksi. In no case should V_s exceed $8\sqrt{f_c'}b_wd$, i.e., $V_n \leq 10\sqrt{f_c'}b_wd$ ($V_w \leq 5\sqrt{f_c'}b_wd$ in WSD).

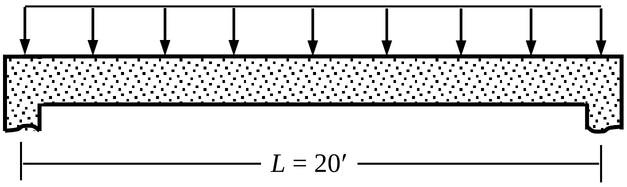
Table 8.1: Summary of ACI Shear Design Provisions (Vertical Stirrups)

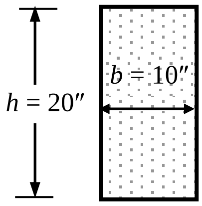
	WSD	USD	Additional Provisions
Design Shear Force	V_w	$V_n = V_u/\phi$ [$\phi = 0.85$]	Calculated at d from Support face
Min^m Section Depth	$V_w/5\sqrt{f_c'}b_w$	$V_n/10\sqrt{f_c'}b_w$	$f_y \leq 60$ ksi
Concrete Shear Strength v_c	$1.1\sqrt{f_c'}$	$1.9\sqrt{f_c'} + 2500\rho_s (Vd/M)$ OR $2\sqrt{f_c'}$	$\sqrt{f_c'} \leq 100$ psi $Vd/M \leq 1.0$
No Stirrup	$V_w \leq V_c/2$	$V_n \leq V_c/2$	
Max^m Spacing	$d/2$, 24″ $S = A_vf_y/50b_w$	$d/2$, 24″ $S = A_vf_y/50b_w$	To be halved if $V_n \geq 6\sqrt{f_c'}b_wd$ OR $V_w \geq 3\sqrt{f_c'}b_wd$ in WSD

Example 8.1

Use the WSD and USD Method to design the vertical stirrups for the simply supported RC beam loaded as shown in (i) and (ii), in addition to self-weight [Given: $f_c' = 3$ ksi, $f_y = 50$ ksi, $f_s = 20$ ksi].

(i) $w_{DL} = 1.0$ k/ft, $w_{LL} = 1.0$ k/ft





In WSD, $v_{cw} = 1.1\sqrt{f_c'} = 60.2$ psi, $v_{cw1} = 3\sqrt{f_c'} = 164.3$ psi, $v_{cw2} = 5\sqrt{f_c'} = 273.9$ psi
and USD, $v_{cu} = 2\sqrt{f_c'} = 109.5$ psi, $v_{cu1} = 6\sqrt{f_c'} = 328.6$ psi, $v_{cu2} = 10\sqrt{f_c'} = 547.7$ psi

Given beam dimensions $b = 10''$, $h = 20''$ $\Rightarrow$ Beam self-weight = $(10 \times 20/12^2) \times 0.15 = 0.208$ k/ft

$\therefore$ Total working load, $w_w = 1.0 + 1.0 + 0.208 = 2.208$ k/ft
Assuming column dimension = $(12'' \times 12'')$ and two layers of bottom steel $\Rightarrow$ Effective beam-depth $d = 20 - 4 = 16''$
$\Rightarrow$ Maximum shear force $V_w = w_w\,(L/2 - c/2 - d) = 2.208 \times \{20/2 - (12/2 + 16)/12\} = 18.03$ k
while $V_{cw} = v_{cw}\,bd = 60.2 \times 10 \times 16 = 9.64$ k, $V_{cw1} = v_{cw1}\,bd = 26.29$ k, $V_{cw2} = v_{cw2}\,bd = 43.82$ k
$\therefore$ Using #3 bars, required stirrup spacing $S = A_v\,f_s\,d/(V_w - V_{cw}) = 0.22 \times 20 \times 16/(18.03 - 9.64) = 8.39''$

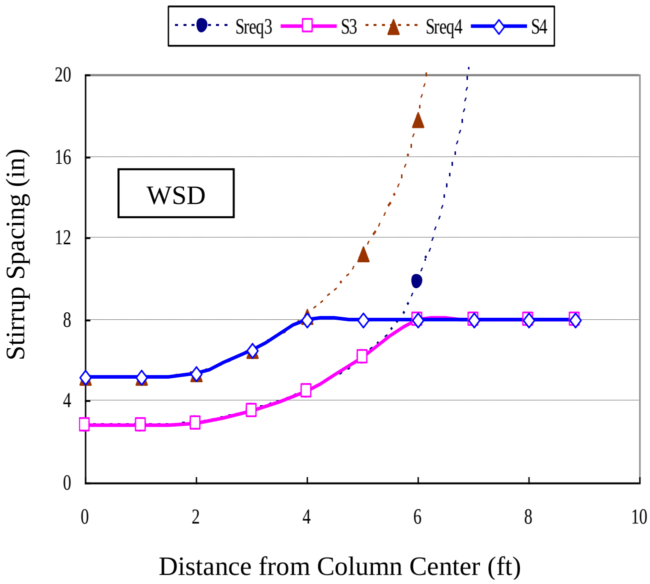
Since $V_w < V_{cw1}$, Minimum stirrup spacing $\leq d/2$ (= 8″), 24″, $A_v\,f_y/(50\,b)$ (= $0.22 \times 50000/500 = 22''$)
No stirrups required if $V_w < V_{cw}/2$ (= 4.82 k); i.e., within (4.82/2.208 =) 2.18′ from center; Else use #3 bars @ 8″ c/c

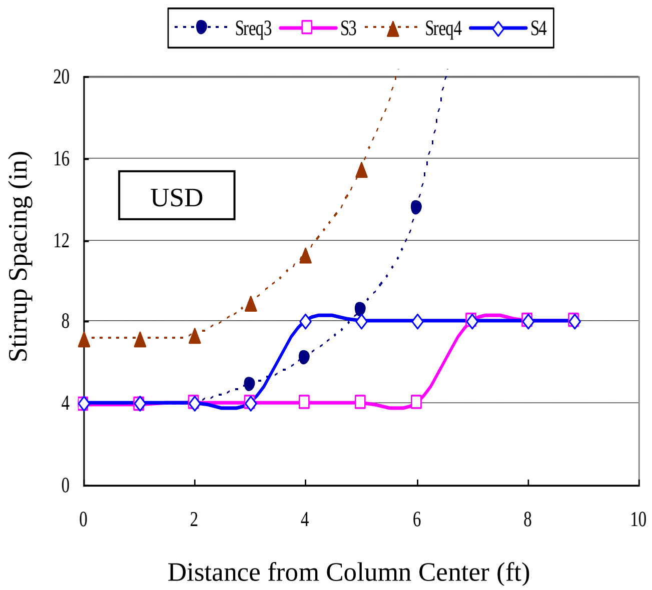
Total ultimate load, $w_u = 1.4 \times 1.0 + 1.7 \times 1.0 + 1.4 \times 0.208 = 3.392$ k/ft
$\Rightarrow$ Maximum shear force $V_u = w_u\,(L/2 - c/2 - d) = 3.392 \times \{20/2 - (12/2 + 16)/12\} = 27.70$ k
and Nominal shear force $V_n = V_u/\phi = 27.70/0.85 = 32.59$ k
while $V_{cu} = v_{cu}\,bd = 109.5 \times 10 \times 16 = 17.53$ k, $V_{cu1} = v_{cu1}\,bd = 52.58$ k, $V_{cu2} = v_{cu2}\,bd = 87.64$ k
$\therefore$ Using #3 bars, required stirrup spacing $S = A_v\,f_y\,d/(V_n - V_{cu}) = 0.22 \times 50 \times 16/(32.59 - 17.53) = 11.69''$
Since $V_n < V_{cu1}$, $S_{min} = 8''$, and no stirrup required if $V_u < V_{cu}/2$ (= 8.77 k); i.e., within (8.77/3.392 =) 2.58′ from center

(ii) If external loads are doubled, i.e., $w_{DL} = w_{LL} = 2.0$ k/ft; Total working load, $w_w = 2.0 + 2.0 + 0.208 = 4.208$ k/ft
$\Rightarrow$ Maximum shear force $V_w = 4.208 \times \{20/2 - (12/2 + 16)/12\} = 34.37$ k
$\therefore$ Using #3 bars, required stirrup spacing $S_{req3} = A_v\,f_s\,d/(V_w - V_{cw}) = 0.22 \times 20 \times 16/(34.37 - 9.64) = 2.85''$
$\therefore$ Using #4 bars, required stirrup spacing $S_{req4} = 0.40 \times 20 \times 16/(34.37 - 9.64) = 5.18''$

Since $V_w > V_{cw1}$, Minimum stirrup spacing $\leq d/4$ (= 4″), 12″, $A_v\,f_y/(50\,b)$ (= $0.40 \times 50000/500 = 40''$)
No stirrups required if $V_w < V_{cw}/2$ (= 4.82 k); i.e., within (4.82/4.208 =) 1.15′ from center

Total load in USD, $w_u = 1.4 \times 2.0 + 1.7 \times 2.0 + 1.4 \times 0.208 = 6.492$ k/ft
$\Rightarrow$ Maximum shear force $V_u = 6.492 \times \{20/2 - (12/2 + 16)/12\} = 53.02$ k, and $V_n = V_u/\phi = 53.02/0.85 = 62.37$ k
$\therefore$ Using #3 bars, required stirrup spacing $S_{req3} = A_v\,f_y\,d/(V_n - V_{cu}) = 0.22 \times 50 \times 16/(62.37 - 17.53) = 3.92''$
$\therefore$ Using #4 bars, required stirrup spacing $S_{req4} = 0.40 \times 50 \times 16/(62.37 - 17.53) = 7.14''$
Since $V_n > V_{cu1}$, $S_{min} = 4''$, and no stirrup required if $V_u < V_{cu}/2$ (= 8.77 k); i.e., within (8.77/6.492 =) 1.15′ from center





Effect of Axial Forces on Shear Strength

Axial Compression

As mentioned before, the concrete shear strength (v_c) is taken as

$$v_c = 1.9\sqrt{f_c'} + 2500\rho_s (V_u d/ M_u) \quad \text{……………………… (8.3)}$$

except that a modified moment

$$M_m = M_u – N_u (4h – d)/8 \quad \text{……………………… (8.13)}$$

is to be substituted for M_u, where h is the total depth of the section and axial force N_u is taken positive for compression. The upper limit of $3.5\sqrt{f_c'}$ is replaced by

$$v_c \leq 3.5\sqrt{f_c'} \sqrt{(1 + N_u/500A_g)} \quad \text{……………………… (8.14)}$$

As an alternative to the rather complicated Eq. (8.3) and (8.13), ACI Code permits the use of an alternative simplified expression

$$v_c = 2\sqrt{f_c'} (1 + N_u/2000A_g) \quad \text{……………………… (8.15)}$$

Axial Tension

For beams subjected to axial tension, the ACI-recommended concrete shear strength is

$$v_c = 2\sqrt{f_c'} (1 + N_u/500A_g) \quad \text{……………………... (8.16)}$$

but not less than zero (N_u is negative for tension). As an alternative, ACI suggests that v_c be taken equal to zero for such members.

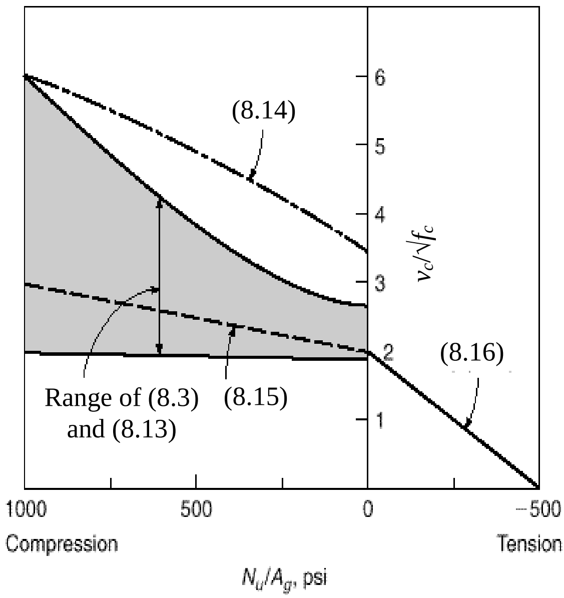


Fig. 8.8: Comparison of equations for v_c for members subject to axial forces

Variation of ($v_c/\sqrt{f_c'}$) with (N_u/A_g) for beams subjected to compression or tension is shown in Fig. 8.8.

Example 8.2

Use the ACI provisions to calculate the concrete shear strength for a beam section with b = 12″, h = 30″, d = 26″, subjected to (i) compressive force N_u = 100 kip, (ii) tensile force N_u = 100 kip, in addition to shear force V_u = 50 kip, M_u = 100 k-ft [Given: f_c' = 4 ksi, f_y = 60 ksi].

Shear Design of Deep Beams

Beams with clear spans L_n less than or equal to 4 times the total member depth h (or $L_n/d \leq 5$) or concentrated loads placed within twice the member depth of a support are called deep beams. In addition to the four components of shear force transfer mentioned before (V_{cz}, V_{iy}, V_d, V_s), a significant part of the shear force is transferred directly from the point of application to supports by 'diagonal struts'. Shear strength of these beams may be even 2 to 3 times greater than that predicted by conventional code equations.

Therefore, the ACI Code provisions permit that the concrete shear strength of such beams can be increased by the factor ($3.5 – 2.5\, M_u/V_u d$) (which is ≥ 1.0 and ≤ 2.5), and computed from

$$v_c = (3.5 – 2.5\, M_u /V_u d)\, (1.9\sqrt{f_c'} + 2500\rho_s V_u d/ M_u) \leq 6\sqrt{f_c'} \quad \text{……………………..}$$

(8.17)

The upper limit of nominal shear strength is specified (perhaps too conservatively) as

$$v_n \leq 8\sqrt{f_c'} \quad \text{for } L_n/d \leq 2 \quad \text{……………………..(8.18a)}$$

and $$v_n \leq 2/3\, (10 + L_n/d)\sqrt{f_c'} \quad \text{for } 2 \leq L_n/d \leq 5 \quad \text{……………………..(8.18b)}$$

Since diagonal failure surface for these beams is inclined at much greater angles than 45°, vertical as well as horizontal web reinforcements take part in resisting shear. When the calculated nominal shear force V_n exceeds the concrete shear force V_c, web reinforcement must carry the excess shear and its contribution is calculated from

$$V_s = V_n – V_c = [A_v/S_v\, (1 + L_n/d)/12 + A_{vh}/S_h\, (11 – L_n/d)/12]\, f_y\, d \quad \text{……………………..(8.19)}$$

where A_v and A_{vh} are the vertical and horizontal shear reinforcements, spaced at S_v and S_h respectively. The code also specifies minimum web reinforcements of

$$A_v \geq 0.0025\, b_w S_v, \text{ where } S_v \leq d/5 \text{ or } 18″ \quad \text{………………..……….(8.20a)}$$

$$A_{vh} \geq 0.0015\, b_w S_h, \text{ where } S_h \leq d/3 \text{ or } 18″ \quad \text{………………..……….(8.20b)}$$

Example 8.3

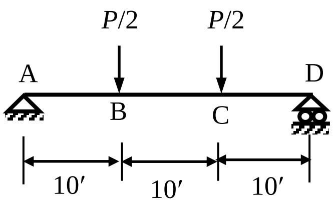
Design the web reinforcements of a 12′ long simply supported beam (with b = 12″) under two-point loading with P_u = 500 kip [Given: f_c' = 4 ksi, f_y = 60 ksi, column size = 18″ × 18″].

Questions and Problems (5)

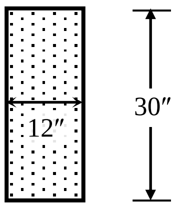
1. (i) Explain Diagonal Tension, Tension Trajectory and Compression Trajectory for flexural shear of beams.
   (ii) Explain the terms Web-Shear Crack and Flexure-Shear Crack.
   Also explain why the Web-Shear Stress is greater than Flexure-Shear Stress.
   (iii) Explain the effects of flexural steel ratio (ρ_s) and (Vd/M) ratio on the allowable shear stress of concrete.
   (iv) What are the components of concrete shear resistance (V_c) of RC beams?
   (v) Explain the effects of Web Reinforcement on the shear resistance of RC beams.
   (vi) Narrate the advantages and disadvantages of inclined stirrups compared to vertical stirrups.
   (vii) Narrate the ACI code provisions for the flexural shear design of beams by WSD and USD.
   (viii) Explain the effect of axial force on the shear strength of concrete.
   (ix) Mention the distinctive features of the shear design of deep beams.

[Use the WSD/USD method for the following problems]

2. (i) Calculate the maximum shear force
   (a) that can be taken by the section without web reinforcement, as well as the ACI Code prescribed shear force without web reinforcement
   (b) the section can possibly take with web reinforcement, as well as the corresponding value of live load P.
   (ii) Design (a) vertical stirrups, (b) 45° inclined stirrups (without considering beam self-weight) for the beam ABCD shown in the figures below under two-point loading if P is equal to the force P calculated in (i)-(b).
   [Given: $f_c' = 4$ ksi, $f_y = 60$ ksi, $f_s = 24$ ksi, column width = 18″].

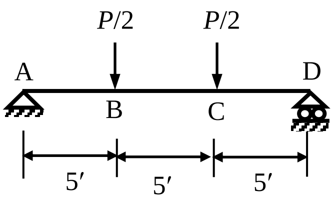


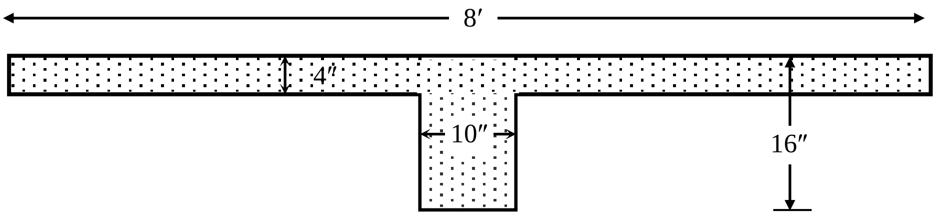
Beam Elevation



Beam Section

3. Answer Question 2 for the beam shown below, considering the slab and beam weights in addition to the applied live loads P [Given: $f_c' = 3$ ksi, $f_y = 40$ ksi, $f_s = 18$ ksi, column width = 12″].



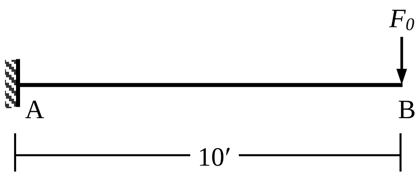


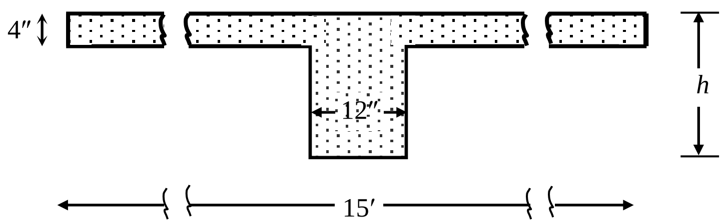
Beam and Slab (c/c) Section

4. Compute the allowable concrete shear forces on the sections described in Question 2 and 3 (at ends and midspans of beams ABCD) reinforced by (i) 6-#8 longitudinal bars for Question 2, (ii) 3-#7 bars for Question 3.

   Compare the forces with the ACI provisions.

5. Design (for shear) the T-section for the cantilever beam AB being subjected to live load $F_0 = 10$ k, floor loads FF = 20 psf, RW = 50 psf, LL = 40 psf in addition to beam/slab self-weights [Given: $f_c' = 3$ ksi, $f_y = 50$ ksi, $f_s = 20$ ksi].

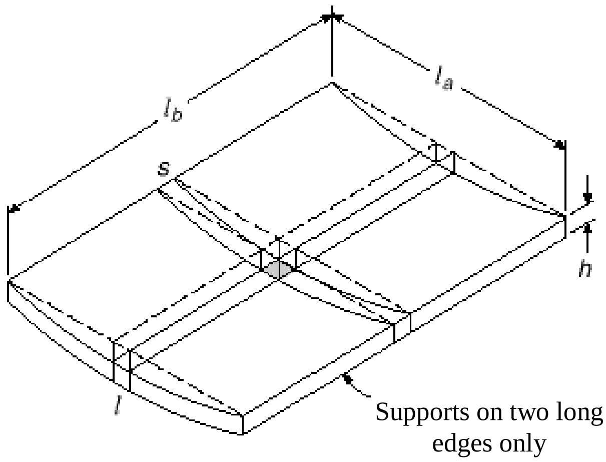




Beam and Slab c/c Section

Design of One-way Slab

Slabs that transfer the imposed loads in one direction only are called one-way slabs. These slabs may be supported on two opposite side only, in which the structural action is essentially one-way, the loads being carried by the slab in direction perpendicular to the supporting beams or walls. There may be beams/walls on all sides, so that two-way action is obtained. If the ratio of length to width is larger than about 2, most of the load is carried in the short direction to the supporting beams and one-way action is obtained in effect, though supports are provided on all sides.



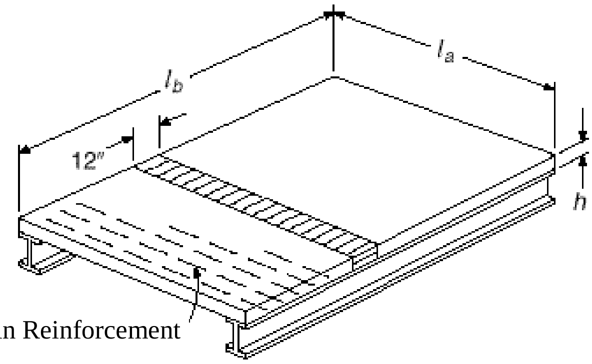


Fig. 9.1: Load transfer and Reinforcements in One-way Slabs (Nilson)

As shown in Fig. 9.1, one-way slabs are analyzed and designed as a series of rectangular beams (of unit strip) cut out at right angles to the supporting beams. The main reinforcements are placed in the direction of the strips, to be spaced less than 18″ or 3 times the slab thickness.

Thickness of One-way Slabs

ACI Code specifies the minimum thickness (t) of one-way slabs (shown in Table 9.1) for non-prestressed slabs of normal weight concrete using Grade 60 reinforcement. The calculations are based on the clear span (L) of the slabs as well as the support conditions; i.e., being larger for comparatively flexible slabs (e.g., cantilever, simply supported) compared to more rigid slabs (e.g., both ends continuous).

Table 9.1: Minimum Thickness of Non-Prestressed One-way Slabs (for f_y = 60 ksi)

Simply Supported	One end continuous	Both ends continuous	Cantilever
$L/20$	$L/24$	$L/28$	$L/10$

Lesser thickness may be used if calculation of deflections indicates no adverse effects.

For reinforcements with $f_y \neq 60$ ksi, the tabulated values are to be multiplied by $(0.4 + f_y/100)$, which accounts for the higher yield strains in high-strength reinforcements, the modulus of elasticity of all grades being essentially the same. The slab thickness (t) is usually rounded to the next higher ¼-in for $t < 6''$ and the next higher ½-in for thicker slabs. ACI recommendation for concrete protection of slab reinforcement is ¾-in (or 1-in from center of steel).

Temperature and Shrinkage Reinforcement

Because of evaporation of excess water after hydration and filling of resulting pours, concrete shrinks as the cement paste hardens. A decrease in temperature may have a similar effect of contraction. As slabs are statically highly indeterminate and joined rigidly to other parts of the structure, they cannot contract freely due to temperature drop or shrinkage. This results in tensile stresses in slabs known as *Temperature and Shrinkage stresses*. Since concrete is weak in tension, these stresses are likely to result in cracking, which should be limited to small and thin cracks known as *hairline cracks*. This can be achieved by placing reinforcements in the slab to counteract and distribute the cracks uniformly. The added steel is known as *Temperature* and *Shrinkage Reinforcement*, or *Distribution Steel*.

ACI recommends the minimum ratios of reinforcements (shown in Table 9.2), which should be placed perpendicular to the main reinforcements, but should not be placed more than 18″ or 5t apart.

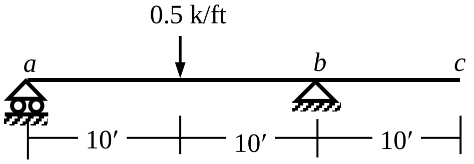
Table 9.2: Minimum Ratios of Temperature and Shrinkage Reinforcement in Slabs

Slabs with f_y = 40 or 50 ksi	0.0020
Slabs with $f_y \geq 60$ ksi	$0.0018 \times (60/f_y) \geq 0.0014$

However, Bangladesh National Building Code (BNBC) suggests at least 50% higher steel ratios (compared to Table 9.2) for concrete made of brick aggregates.

Example 9.1

Design the RC slab shown below supported on 10″ thick walls (carrying FF = 30 psf, RW = 50 psf + 0.5 k/ft at midspan of *ab*, LL = 60 psf, in addition to self weight) using design moments [Given: f_c' = 3 ksi, f_y = 50 ksi].



Clear span of *ab* = 20 – 10/12 = 19.17′, and of *bc* = 10 – 5/12 = 9.58′

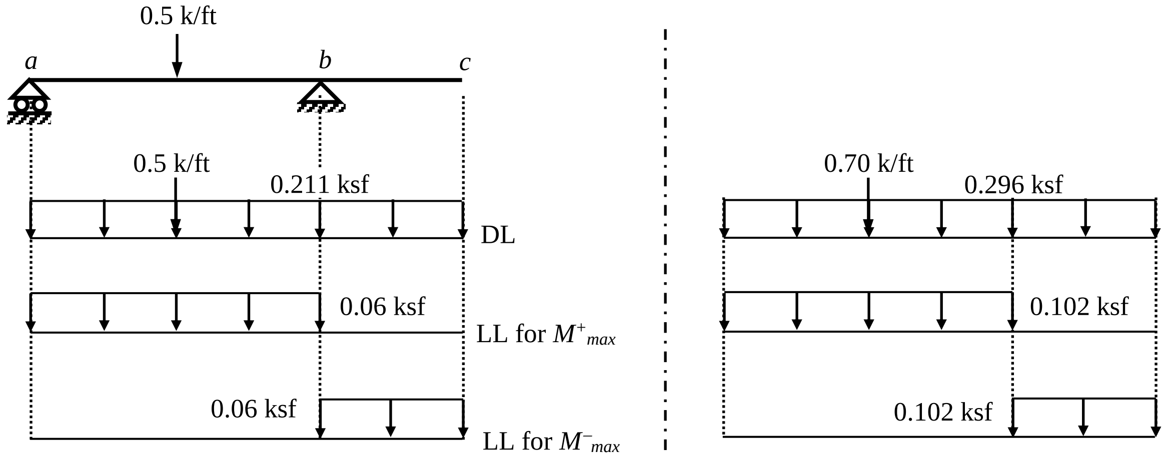
∴Required thicknesses are t_{ab} = 19.17 × 12/24 = 9.58″, and t_{bc} = 9.58 × 12/10 = 11.5″

∴Minimum thickness for f_y = 50 ksi is t = 11.5 × (0.4 + 50/100) = 10.35″

∴Thickness of 10.5″ (d = 9.5″) is taken throughout the slab ⇒ Self-weight = 10.5/12 × 150 = 131.25 psf

∴Total DL = (131.25 + 30 + 50 =) 211.25 psf + 0.5 k/ft and LL = 60 psf

Loading situations are



M^+_{max} = 10.92 k′/′ | M^-_{max} = 13.56 k′/′

k = 0.378, j = 0.874, R = 0.223 ksi

d_{req} = √(13.56/0.223) = 7.80″< 9.5″, OK

$A_s{}^+{}_{max}$ = 10.92 × 12/(20 × 0.874 × 9.5) = 0.79 in²/′

$A_s{}^-{}_{max}$ = 13.56 × 12/(20 × 0.874 × 9.5) = 0.98 in²/′

A_{stemp} = 0.0025*bt* = 0.0025 × 12 × 10.5 = 0.315 in²/′

[V_{max} = 3.64 k/′ (from structural analysis)

v_{max} = 3.64 × 1000/(12 × 10.5) = 28.9 psi < 1.1√f_c']

M^+_n= 18.82 k′/′ | M^-_n = 23.41 k′/′

For M^+_n, a = 9.5 [1–√{1–2×18.82/(2.55× 9.5²)}] = 0.81″

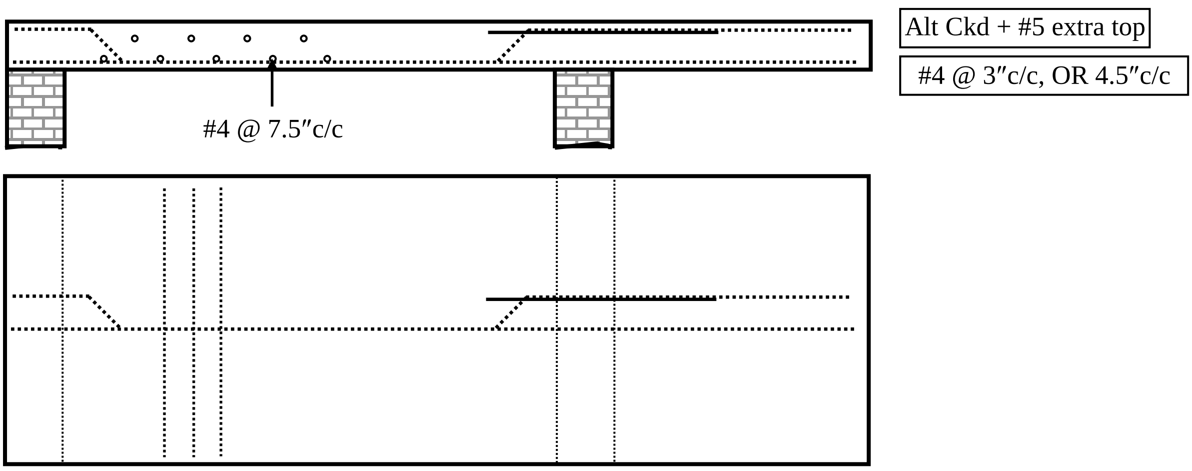
$A_s{}^+{}_{max}$ = 2.55 × 0.81 × 12/50 = 0.50 in²/′

For M^+_n, a = 9.5 [1–√{1–2×23.41/(2.55× 9.5²)}] = 1.02″

$A_s{}^+{}_{max}$ = 2.55 × 1.02 × 12/50 = 0.63 in²/′

[V_n = 6.26 k/′ (from structural analysis)

v_{max} = 6.26 × 1000/(12 × 10.5) = 49.7 psi < 2√f_c']

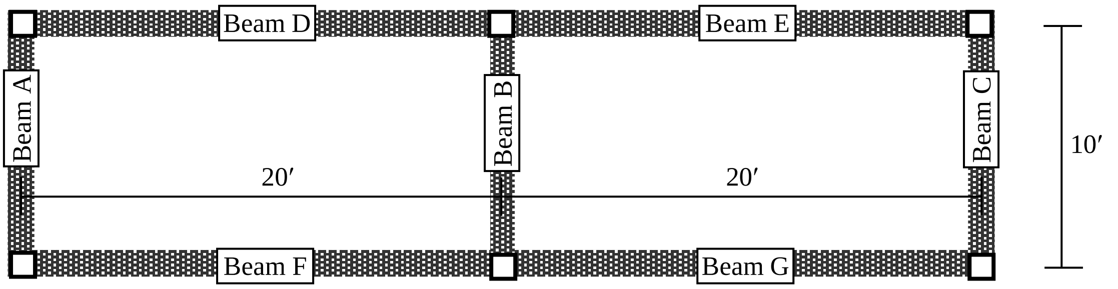


Side Elevation and Plan by WSD and USD

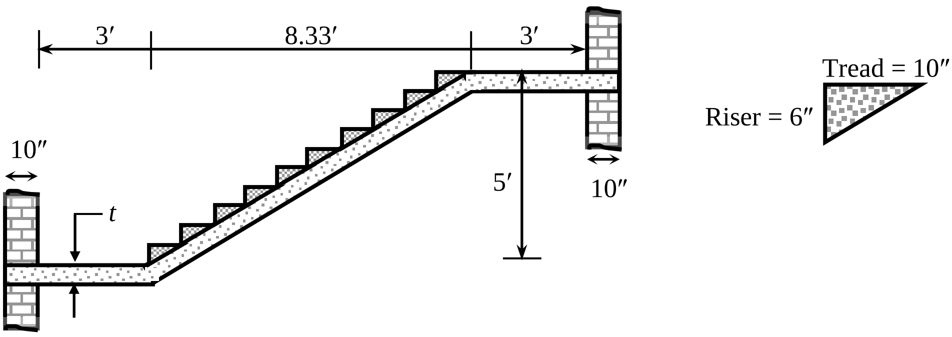
Questions and Problems (6)

1. (i) What is one-way slab? Give some common examples of one-way slabs in engineering structures.
   (ii) Narrate the ACI code provisions for choosing the minimum thickness of one-way slabs.
   Explain why the required thickness of one-way slabs increase with the yield strength of reinforcing steel.
   (iii) Narrate the necessity and ACI code provisions for temperature and shrinkage reinforcement in slabs.
   Explain why the minimum flexural steel required by beams is not applicable for slabs.
   (iv) Explain why shear reinforcements are usually not provided in the design of RC slabs.

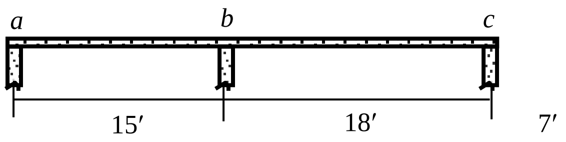
2. In the floor system shown below, calculate required slab thickness and reinforcements if
   (i) Beams D, E, F, G are removed (i.e., slabs are supported on Beams A, B, C),
   (ii) Beam B is removed (i.e., slabs are supported on Beams A, C, D, E, F, G),
   (iii) Beams A, C, D, E, F, G and corner columns are removed (i.e., slabs are supported on Beam B).



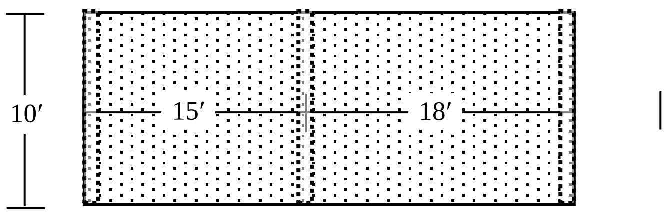
3. Figure below shows a staircase simply-supported on 10″ brickwalls.
   Determine the thickness (t) of the waist-slab and use the WSD to calculate the
   (i) allowable live load on the staircase if FF = 20 psf,
   (ii) required reinforcements in the slab [and show them with neat sketch].



4. Design the RC slab *abc* shown below (supported on 12″ wide beams and carrying FF = 30 psf, RW = 50 psf, LL = 60 psf, in addition to self weight) using ACI moment and shear coefficients [Given: f_c' = 4 ksi, f_y = 60 ksi].

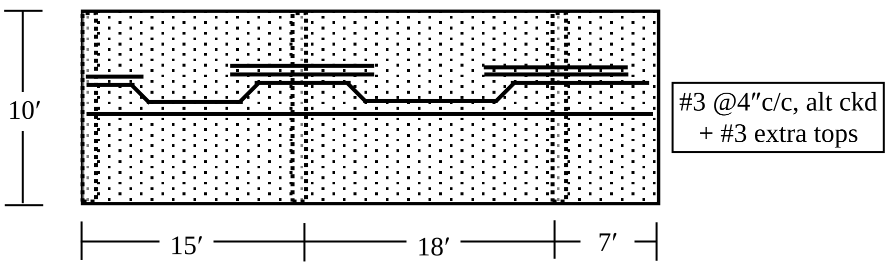


Side Elevation



Plan

5. Calculate the live load that can be carried by the slab shown below (carrying FF = 30 psf, RW = 50 psf), if it is reinforced (longitudinally) as shown below. Also show its reinforcements in the transverse direction [Use ACI moment coefficients, with f_c' = 4 ksi, f_y = 60 ksi].



Bond, Anchorage and Development Length

Bond Force

If the reinforced concrete beam of Fig. 10.1(a) is loaded as shown in Fig. 9.1(b), its longitudinal tension reinforcing bars would tend to maintain their original length and slip longitudinally with respect to the original beam.

(a) (b)

Fig. 10.1: Beam before and after loading (Nilson)

In order for RC to behave as intended, it is essential that bond forces develop on the interface between concrete and steel to prevent significant slip from occurring at the interface. Therefore, reinforcing bars would be subject to forces shown in Fig. 10.2(a), while the bars would apply forces on the surrounding concrete, as shown in Fig. 10.2(b).

(a) (b)

Fig. 10.2: Bond forces on steel and concrete (Nilson)

Bond Failure

Two types of bond failure is observed in reinforcing bars in tension

(i) *Splitting* of concrete itself along the bars (along vertical or horizontal plane), when the concrete cover, confinement or bar spacing is insufficient to resist tension (Fig. 10.3).

(ii) *Direct Pullout* of the bars, occurring when ample confinement is provided by the surrounding concrete; e.g., when relatively slender bars are surrounded by sufficiently large concrete covers and bar spacing.

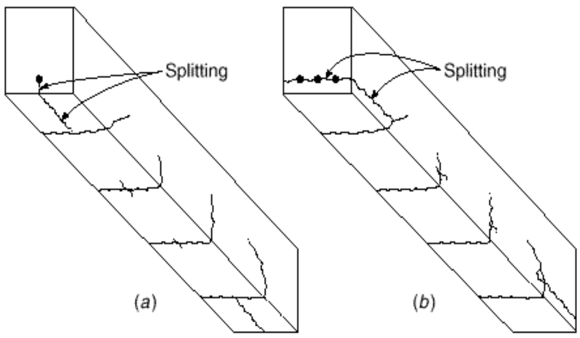


Fig. 10.3: Splitting of Concrete along Reinforcement (Nilson)

Pullout, Anchor and Tied-Arch Action

For plain reinforcing bars, the initial bond strength is provided by weak chemical adhesion and mechanical friction between steel and concrete, followed by some natural interlocking of the bars with concrete. However, this natural bond strength is low and frequently broken, resulting in bars being pulled through the concrete. End anchorage is provided in such bars in the form of hook (Fig. 10.4), to prevent beam failure even if the bond is broken over the entire length of the beam. This is so because the member acts as a tied arch.

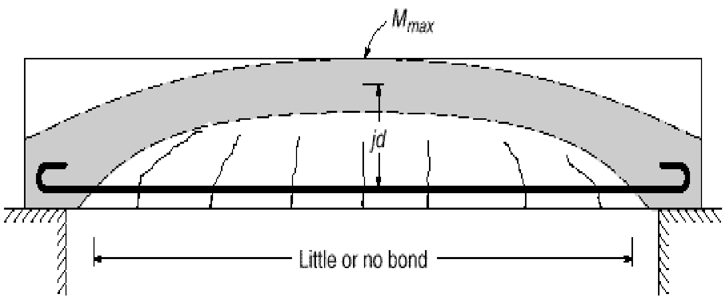


Fig. 10.4: Tied-arch action in beam with little bond (Nilson)

Development Length

Development length of a reinforcing bar is defined as that length of embedment necessary to develop the full tensile strength of the bar, controlled by splitting or pullout. In Fig. 10.5, the steel stress is zero at the support and maximum at point *a*. Therefore, the maximum tensile force A_sf_y must be developed within the length 'l' in order to prevent bond failure of the beam. The length over which this force develops is called Development length.

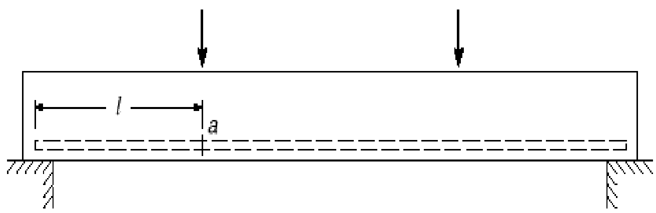


Fig. 10.5: Development Length of Reinforcing Bar (Nilson)

Factors Affecting Development Length

The following factors have been found to influence the development length of reinforcing bars

* Tensile strength of concrete: The most common type of bond failure is splitting of concrete, associated with tension failure of concrete. Therefore, l_d is inversely proportional to tensile strength of concrete. Also since tensile strength of lightweight concrete is less than for normal-density concrete, development lengths must be increased for them.
* Cover distance and Bar spacing: If the vertical and horizontal distance (clear cover or bar spacing) is increased, more concrete is available to resist the tension in concrete, decreasing the development length. For example, the bar spacing for slabs are typically much more for beams, resulting in reduced l_d.
* Lateral reinforcement: Provided by the stirrups, they tend to check the opening and propagation of cracks. Their effectiveness depends on the strength, area and spacing along the development length.
* Vertical bar location: If bars are placed in beams with a substantial depth of concrete placed below, there is a tendency for excess water and entrapped air to rise to the top during vibration. Accumulating under the bars, they tend to decrease the bond strength and increase development length of the bars.
* Epoxy coating of bars: They are often used to prevent corrosion of reinforcing bars (e.g., in bridge decks and parking garages), reducing the bond-strength by preventing the adhesion between concrete and steel. The development length of such bars increase as a result.
* Bar diameter: Smaller diameter bars require somewhat less development length than predicted by empirical equations developed for larger bars.
* Over design of reinforcing bars: Tensile reinforcements are often provided in excess of the calculated amount required. Such over designs reduce the stress in the bars provided, reducing the development length of such bars.

ACI suggests the following formula for development length of deformed bars in tension (in terms of bar diameter d_b)

$$l_d/d_b = (3/40)\ (f_y/\sqrt{f_c'})\ (\alpha\beta\gamma\lambda)/\{(c + K_{tr})/d_b\} \qquad \text{.......... (10.1)}$$

where the term $(c + K_{tr})/d_b$ is ≤ 2.5. The terms in Eq. (10.1) are defined in Table 10.1

Table 10.1: Parameters of Development Length of Tension Bars

Symbol	Parameter	Variable	Value
α	Reinforcement Location Factor	* Horizontal Reinforcement over ≥ 12″ concrete * Other Reinforcement	1.3 1.0
β	Coating Factor	* Epoxy-coated bars with cover <3d_b or clear spacing <6d_b * All other epoxy-coated bars or wires * Uncoated bars * Maximum value of $\alpha\beta$	1.5 1.2 1.0 1.7
γ	Reinforcement Size Factor	* ≥ #7 bars * ≤ #6 bars and deformed wires	1.0 0.8 (?)
λ	Lightweight Aggregate Concrete Factor	* When lightweight aggregate concrete is used * When normal-weight concrete is used	1.3 1.0
c	Spacing or Cover Dimension (in)	* Bar center to nearest concrete cover * One-half the c/c spacing of bars	Smaller than both
K_{tr}	Transverse Reinforcement Index	S = Maximum spacing of transverse reinforcement A_{tr} = Area of all transverse reinforcement within S f_{tr} = Yield strength of transverse reinforcement, ksi n = No. of bars being developed along the plane of splitting	$A_{tr}\ f_{tr}/(1.5Sn)$

Table 10.2: Simplified Equations for Basic Development Length (Tension)

Condition	$(c + K_{tr})/d_b$	l_d
Avoid pullout failure (Experimentally derived limit)	2.5	$0.03\ (f_y/\sqrt{f_c'})d_b$
* Clear cover and Clear spacing ≥ d_b + Code required stirrups * Clear cover and Clear spacing ≥ 2d_b	1.5	$0.05\ (f_y/\sqrt{f_c'})d_b$ (≥ #7 Bars) $0.04\ (f_y/\sqrt{f_c'})d_b$ (≤ #6 Bars and deformed wires) (?)

Besides, mode of load transfer (tension/compression), presence of end anchorage influences development lengths.

* Bars subjected to compression transfer loads in the form of bond along its length (which is also less cracked) as well as end bearing, so smaller development lengths are required for bars in compression. This is reduced even further by the presence of transverse confinement like spiral reinforcements.
* Hooks prevent bond failures by providing end anchorage of bars with concrete, and therefore hooked bars require smaller development lengths.

Table 10.3: Development Lengths for Deformed Bars in Compression

Basic Development Length	$0.02\,(f_y/\sqrt{f_c'})d_b$ but $\geq 0.0003\,f_y\,d_b$
Modification factors	
* Reinforcement excess of that required by analysis	$A_{s(required)}/A_{s(provided)}$
* Reinforcement enclosed within spiral (or #4 ties spaced @≤4″c/c)	0.75

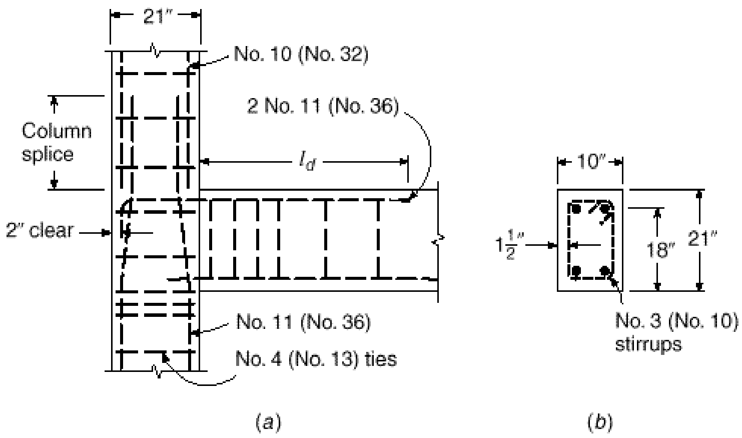
Table 10.4: Development Lengths for Hooked Deformed Bars in Tension

Basic Development Length	$0.02\,(f_y/\sqrt{f_c'})d_b$
Modification factors	
* Reinforcement excess of that required by analysis	$A_{s(required)}/A_{s(provided)}$
* β: Epoxy-coated bars	1.2
* λ: Lightweight aggregate concrete	1.3
* Side cover ≥ 2.5″, end cover ≥ 2″	0.7
* Enclosed within ties or stirrups	0.8

<u>Example 10.1</u>

A beam section (b = 10″, h = 21″, d = 18″, clear cover = 1.5″), made of normal-density concrete with f_c' = 4 ksi, is reinforced with 2-#11 bars (d_b = 1.41″, A_s = 3.12 in^2, with f_y = 60 ksi), whereas the reinforcement required from structural analysis is 2.90 in^2, in addition to 4-#3 stirrups @3″c/c, followed by #3 stirrups @5″c/c.

Calculate the development length l_d of the bars, using (i) Table 10.1, and (ii) Table 10.2.



The factors $\alpha, \beta, \gamma, \lambda$ in Table 10.1 are

α = 1.3 (Reinforcement over ≥ 12″ concrete)

β = 1.0 (Uncoated bars)

γ = 1.0 (≥ #7 bars)

λ = 1.0 (Normal-weight concrete)

$\therefore \alpha\beta\gamma\lambda = 1.3 \times 1.0 \times 1.0 \times 1.0 = 1.3$

(i) Bar center to nearest concrete cover

Top = 3″, Side = 1.5 + 0.38 + 1.41/2 = 2.58″

c/c spacing of bars = 10 – 2 × 2.58 = 4.84″

$\therefore c$ is the smallest of 3″, 2.58″ and (4.84/2 =) 2.42″

$K_{tr} = A_{tr}\,f_{tr}/(1.5Sn) = 0.22 \times 60/(1.5 \times 5 \times 2) = 0.88''$

$\therefore (c + K_{tr})/d_b = (2.42 + 0.88)/1.41 = 2.34 < 2.5$

$\therefore l_d = (3/40)\,(f_y/\sqrt{f_c'})\,d_b\,(\alpha\beta\gamma\lambda)/\{(c + K_{tr})/d_b\} = (3/40) \times [60000/\sqrt{(4000)}] \times 1.41 \times (1.3)/2.34 = 55.7''$

⇒ Required development length = 55.7 × (2.90/3.12) = 51.8″

(ii) Minimum clear cover = 2.58 – 1.41/2 = 1.88″and Clear spacing between bars = 4.84 – 1.41 = 3.43″

Both are > d_b, and code-required stirrups are also provided.

$\therefore$ Table 10.2 formulae can be applied; with $(c + K_{tr})/d_b = 1.5$;

i.e., $l_d = (0.05)\,(f_y/\sqrt{f_c'})\,d_b\,(\alpha\beta\gamma\lambda) = (0.05) \times [60000/\sqrt{(4000)}] \times 1.38 \times (1.3) = 86.9''$

⇒ Required development length = 86.9 × (2.90/3.12) = 80.8″

Anchorage of Bars

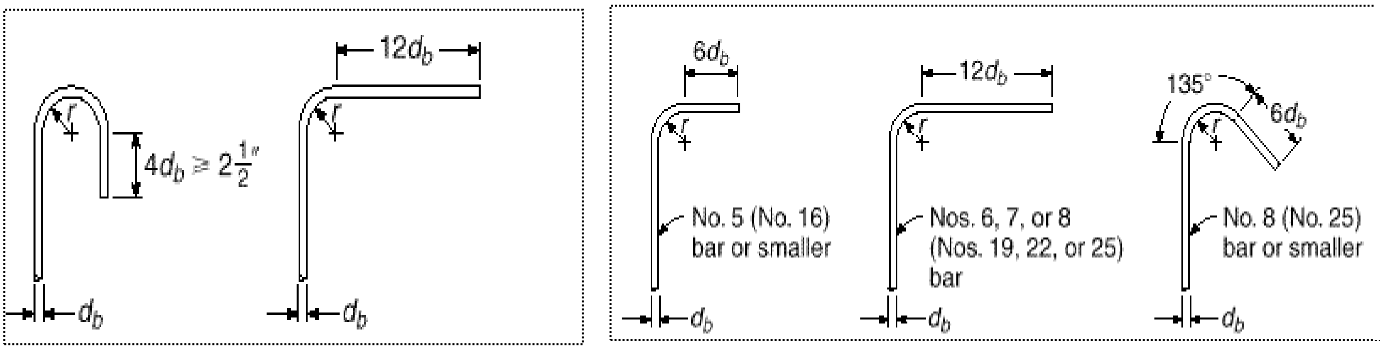


Fig. 10.6: Standard Bar Hooks: (a) Main Reinforcement, (b) Stirrups and Ties

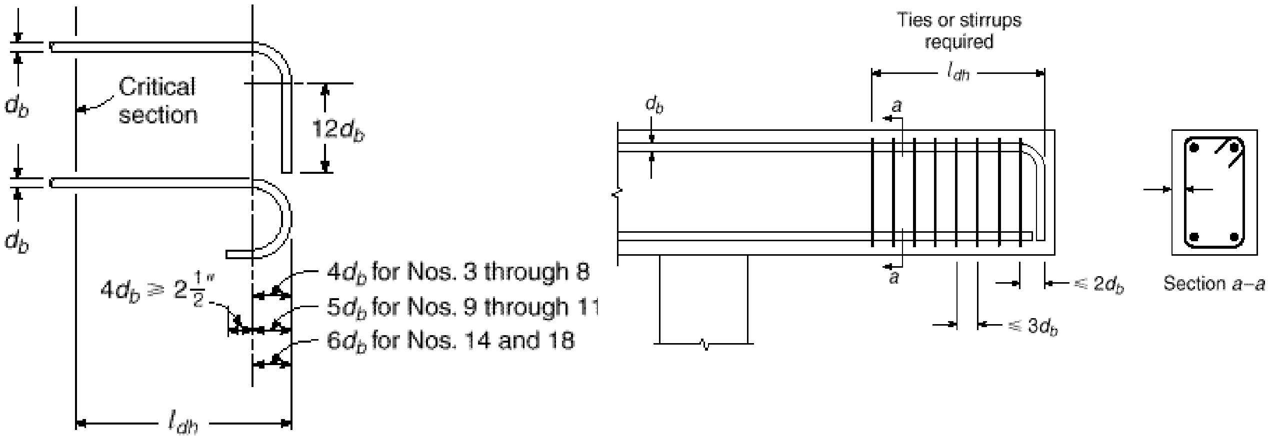


Fig. 10.7: (a) Bar Details for development of standard hooks, (b) Lateral reinforcement requirements at discontinuous ends

Bundled Bars
It is often advantageous to bundle bars in groups of 2, 3 (triangular or L-shaped) or 4 (square) in large beams for improved placement of concrete around the bundled bars. Development lengths of bundled bars are larger (typically 20% larger for 3 bars and 33% larger for 4 bars) than the individual bars, to account for the probable deficiency of bond within the bundle.

Lap Splice
Typically reinforcing bars are stocked by suppliers in specified limited lengths (e.g., of 20~60ft). Since it is often more convenient to work with shorter bar lengths, bars are often spliced on site. Splicing of bars at maximum stress should be avoided, and they should be staggered when used.

Two different types of lap splices are used for *bars in tension*, corresponding to minimum length of lap.

Class A: Lap of l_d, provided when area of reinforcement is at least twice as area required by analysis, over the entire length of the lap length.

Class B: Lap of $1.3l_d$, when Class A is not provided.

Compression bars (mainly in columns) may be spliced by lapping, by direct end bearing or by welding or mechanical devices. Minimum lap length for compression splices is set by ACI as

$0.5f_y\, d_b$ $[f_y \leq 60 \text{ ksi}]$

and $(0.9f_y - 24)\, d_b$ $[f_y > 60 \text{ ksi}]$

Minimum lap length of compression bars in 12″.

Bar Cut-Off and Bend Points in Beams

Bars calculated from the design of critical beam sections can be cut-off or bent over the length of the beam. Such curtailment or re-adjustment of bars may result in significant economy of design, particularly in long span beams.

The tensile force to be resisted by the reinforcement at any cross-section is $T = A_s f_s = M/z$, where M is the bending moment at the section and z is the internal moment arm, which varies only within narrow limits and is never less than z at maximum moment section. Therefore, the tensile force and required steel area is almost proportional to M.

Fig. 10.8(a) and (b) are the steel-requirement diagrams for simply supported and continuous beams under UDL, showing theoretical bar cut-off points from the moment diagrams.

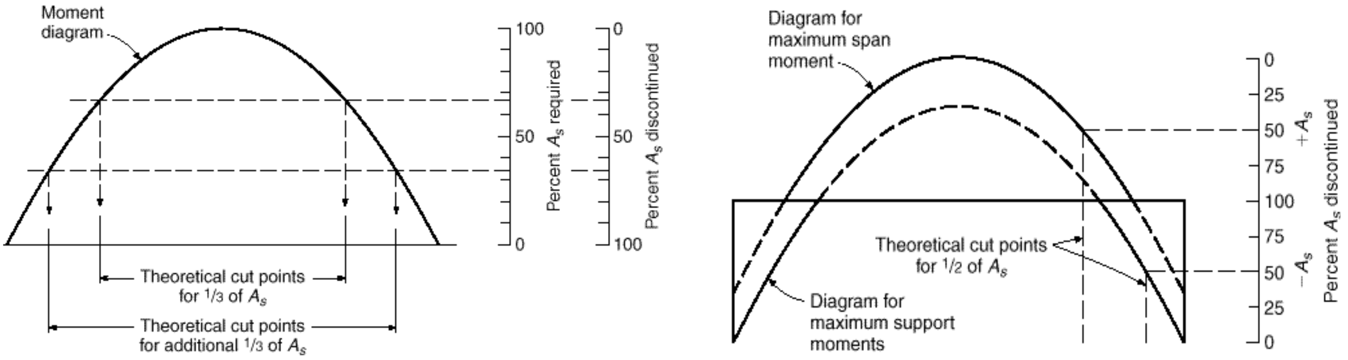


Fig. 10.8: Bar Cut-Off Points from BMD: (a) Simply Supported Beam, (b) Continuous Beam (Nilson)

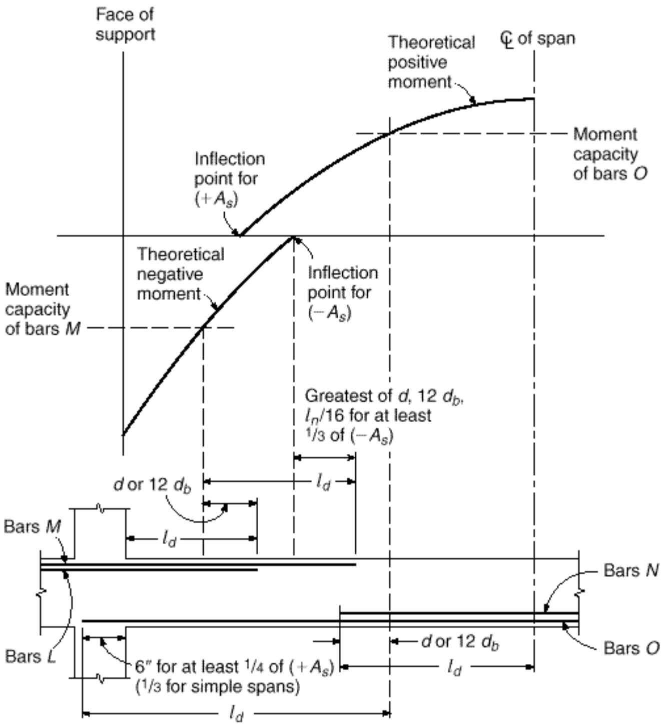


Fig. 10.9: Bar cutoff requirements of ACI Code (Nilson)

However, in no case should the tensile steel be discontinued exactly at the theoretically described points, because of

* Propagation of diagonal tension cracks
* Difference between design and actual moment diagrams due to approximations of loads, analysis

ACI Code requires

* Each bar should be continued at least a distance equal to d or $12d_b$ (whichever is larger) beyond where it is theoretically not required.
* Full l_d must be provided beyond critical sections
* A minimum portion of $A_s^{(+)}$; i.e., 1/3 (in simple spans) and 1/4 (in continuous spans) must be continued at least 6″ into the support, and anchored if there is possibility of load reversal.
* At least $A_s^{(-)}/3$ must be extended d or $12d_b$ or L/16 (whichever is larger) beyond point of inflection.

Fig. 10.9 summarizes ACI requirements for bar cut-off.

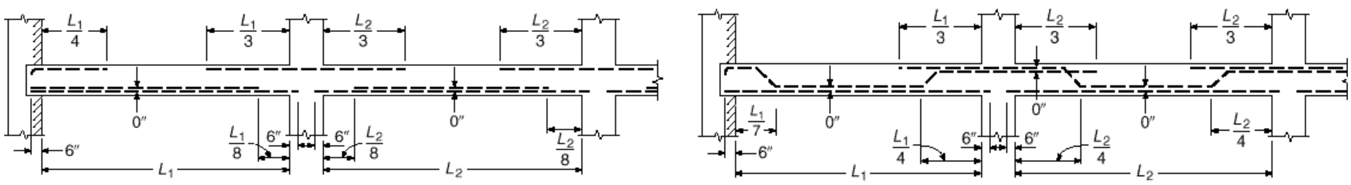


Fig. 10.10: Standard (a) cutoff and (b) bend points for bars in nearly equal spans with UDL (Nilson)

As the determination of bar cut-off and bend points may be rather tedious, particularly for statically indeterminate beams, many designers specify bar cut-off and bend points at more or less arbitrarily defined points that experience has proven to be safe. The standard cut-off and bend points shown in Fig. 10.10 have been found to be satisfactory for approximately equal spans under UDL.

Example 10.2

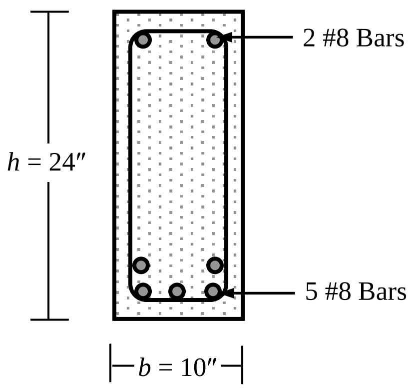
Simply supported beams (of 20′ span) are loaded by
(a) uniformly distributed load of 2.0 k/ft, (b) concentrated load of 20 k at midspan.

Design the beams and show the bar cut-offs using ACI detailing recommendations [Given: $f_c' = 3$ ksi, $f_y = 50$ ksi].

For $f_c' = 3$ ksi = 3000 psi, $n = 9$, $f_{call} = 1.35$ ksi, $f_{sall} = 20$ ksi $\Rightarrow r = 14.81$

$\therefore k = 0.378$, $j = 1 - k/3 = 0.874$, $R = 0.5\, f_{call}\, kj = 0.223$ ksi

In both cases (a) and (b), the maximum bending moment due to applied load is 100 k-ft [i.e., $= 2 \times 20^2/8 = 20 \times 20/4$]
In addition, assuming $b = 10''$ and $h = 24''$, beam self-weight $= (10 \times 24/12^2) \times 0.15 = 0.25$ k/′
$\therefore$ Additional moment due to self-weight $= 0.25 \times 20^2/8 = 12.5$ k′

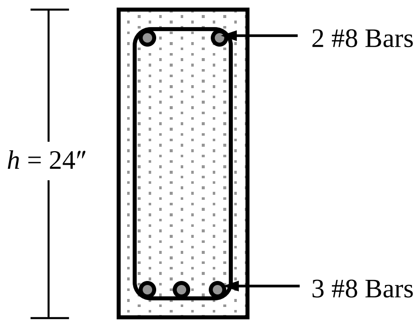


$M_{max} = 100 + 12.5 = 112.5$ k′
$\therefore$ Assuming two layers of steel, $d = 24 - 4 = 20''$
$\therefore M_1 = Rbd^2 = 0.223 \times 10 \times 20^2/12 = 74.32$ k′
$\Rightarrow M_2 = M - M_1 = 38.18$ k-ft

$\therefore A_s = A_{s1} + A_{s2} = M_1/(f_s\, jd) + M_2/[f_s\,(d-d')]$
$= 74.32 \times 12/(20 \times 0.874 \times 20) + 38.18 \times 12/[20 \times 17.5]$
$= 2.55 + 1.31 = 3.86$ in^2; i.e., 5 #8 Bars in two layers

Also $f_s' = 2 \times 20 \times (0.378 - 2.5/20)/(1-0.378) = 16.26$ ksi
$\Rightarrow A_s' = 38.18 \times 12/[16.26 \times (20-2.5)] = 1.61$ in^2;
i.e., 2 #8 Bars in one layer

This is a repeat of the design shown in Example 5.1



The allowable moment in the section after cutting off 2 bottom bars can be calculated by analyzing the section.

Cutting 2 bottom bars $\Rightarrow A_s = 2.37$ in^2, one steel layer $d = 24-2.5 = 21.5''$
$\rho_s = 2.37/(10 \times 21.5) = 0.011$, $n\rho_s = 9 \times 0.011 = 0.099$

Neglecting compression bars $\Rightarrow k = \sqrt{\{2n\rho_s + (n\rho_s)^2\}} - n\rho_s = 0.357$,
$\therefore j = 0.881$, $R = 1.35 \times 0.357 \times 0.881/2 = 0.212$ ksi
$\therefore M_c = Rbd^2 = 0.212 \times 10 \times 21.5^2/12 = 81.81$ k′
$\therefore M_s = A_s f_s jd = 2.37 \times 20 \times 0.881 \times 21.5/12 = 74.81$ k′

$\therefore M = 74.81$ k′, which is close to $112.5 \times 3/5 = 67.50$ k′

This is (112.5–74.81 =) 37.69 k′ less than the maximum moment, which is the moment at a distance x from midspan

In (a), $2.25\, x^2/2 = 37.69 \Rightarrow x = 5.79' = 69.5''$

and in (b), this is derived from $12.5\, x_0 - 0.25\, x_0^2/2 = 74.81 \Rightarrow x_0 = 6.39' \Rightarrow x = 3.61' = 43.3''$

ACI recommends extensions beyond the theoretical cut-off points not less than d (= 21.5″), or $12d_b$ (= 12″)

Moreover, the bars can only be cut-off a distance $l_d \cong 0.05 \times 50000/\sqrt{(3000)} \times 1.0 = 45.6''$ from the critical section.

This is satisfied comfortably by cutting the bars at a distance (from center)
(= 69.5 + 21.5) = 91.0″ in (a), and (= 43.3 + 21.5) = 64.8″ in (b)

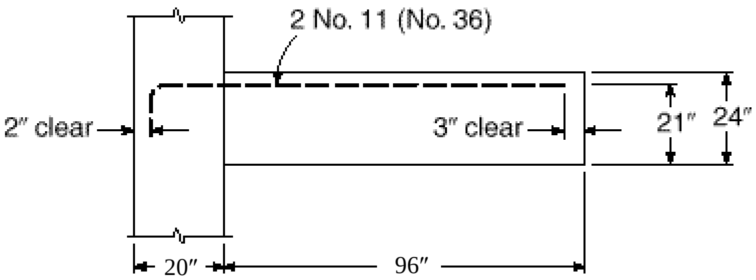
Another bar (central #8) can be curtailed similarly, at (= 88.8 + 21.5) = 110.3″ from center in (a) and (= 69.0 + 21.5) = 90.5″ in (b), but the other two should be continued at least 6″ into the columns.

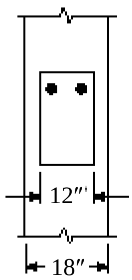
Questions and Problems (7)

1. (i) What is development length? Mention the factors influencing development length of deformed bars in tension.
   (ii) Briefly compare between the development lengths of
   (a) Bottom and top bars, (b) Epoxy-coated and uncoated bars, (c) Large diameter and small diameter bars.
   (iii) Explain the effects of concrete tensile strength, cover distance and lightweight aggregate on the development length in tension.
   (iv) Explain why the development length of compression bars is smaller than that of tension bars.
   (v) Narrate the necessity of hook and anchorage for plain and deformed bars.
   (vi) What are bar splices? Distinguish between lap splices in tension and compression.
   (vii) Explain why flexural reinforcement bars are not cut off exactly where they are not theoretically required. Also justify the difference among the approximate bar cut-off points (at $L_n/3$, $L_n/4$, $L_n/7$, $L_n/8$ from supports).

2. The tensile flexural reinforcement required in the cantilever beam shown below is A_s = 2.80 in^2, which is provided by two #11 bars (for d = 21″), while #3 transverse reinforcements with 1.5″ cover are provided starting at 4″ from column face, with 3 @ 8″c/c and 5 @ 10″c/c.

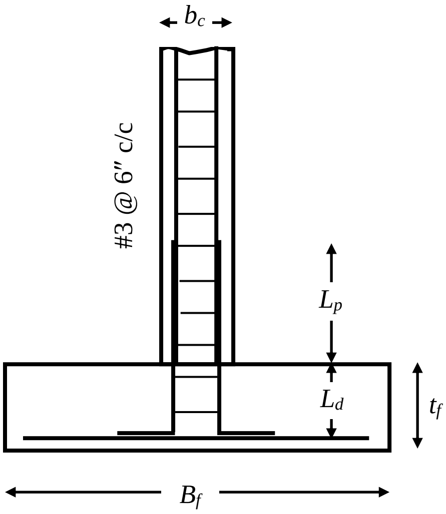
   Check if the #11 bars (shown in figure below) are provided adequate
   (a) development length in the beam, (b) embedment within the column [Given: f_c' = 4 ksi, f_y = 50 ksi].





3. Rework Question 2 using 4 #8 (instead of 2 #11) bars.

4.



Calculate the
(a) Development length L_d and lap splice L_p for the column-footing connection (under compression) shown on left, if all the column bars are #7.

(b) Minimum required thickness (t_f) of the footing

(c) Maximum footing bar size if width (B_f) of the footing is 6′

[Given: b_c = 18″, f_c' = 4 ksi, f_y = 50 ksi].

5. Simply supported beams (of 20′ span) are loaded by

   (a) uniformly distributed load of 2.0 k/ft, (b) concentrated load of 20 k at midspan.

   Design the beams and show the bar cut-offs using ACI detailing recommendations [Given: f_c' = 3 ksi, f_y = 50 ksi].

Following are some of the theoretical questions discussed after midterm exam, with guidelines for their answers.

- The examination questions can be different or mixed or parts (based on the same topics/concepts).
- Don't copy this language, just follow the points and read books to prepare your own answers.

1. Mention the differences and advantages of designing beams as T-beams compared to rectangular beams.
   - Effective width (b_{eff}) in T-beam determined from three conditions.
   - Large b_{eff} provide large compression area, therefore compression rarely governs in T-beams. Also smaller depths may be provided because of the larger b_{eff}.
2. Explain the differences between flexural stress distribution over T- and rectangular beams (and their effects).
   - Two-dimensional stress distribution due to tension/compression from slab (effect on f_c').
   - Variation of stresses across the width of beam due to shear deformations (effect on b_{eff}).
3. Mention and explain the ACI-recommended effective widths of T- and L-Beams.
   - Variation of stress across the width of the beams.
   - Mention the T-beam widths and L-beam widths.
4. Explain Diagonal Tension, Tension Trajectory and Compression Trajectory for flexural shear of beams.
   - Shear failure of RC, is more properly called *Diagonal Tension Failure* due to the mode of failure, i.e., as the principal tensile stress that acts diagonally (provide Figures).
   - Tension Trajectory is the curve joining the directions of maximum diagonal tensile stress in a beam, while Compression Trajectory is the curve joining the directions of maximum compressive stress.
5. Explain Web-Shear crack and Flexure-Shear crack. Also explain which is greater.
   - Web-Shear crack is due to diagonal tension from maximum shear.
   - Flexure-Shear crack is due to diagonal tension following initial crack from maximum moment.
   - Flexure-Shear crack is smaller because the cracked (and smaller) section involved.
6. Explain the effects of flexural steel ratio (ρ_s) and (Vd/M) ratio on the allowable shear stress of concrete.
   - Flexural steel leads to smaller and narrower flexural cracks, leaving a larger uncracked concrete area to resist shear, and increasing shear resistance of section.
   - Stress where diagonal cracks develop depends on ratio of shear and flexural stress, i.e., proportional to Vd/M.
   - Mention the formula with upper limit representing web-shear crack.
7. Explain the effects of Web Reinforcement on the shear resistance of RC beams.
   - Mention concrete shear resistance components due to uncracked concrete (V_{cz}), interlocking across crack (V_{iy}) and dowel action (V_d).
   - Web Reinforcements enhance beam shear resistance after the formation of diagonal cracks, by
– restricting lengthening of diagonal cracks into concrete compression zone (thereby increasing V_{cz}),
– restricting widening of cracks (thereby increasing the interface force V_{iy}),
– confining the longitudinal bars (thereby resisting their splitting and increasing dowel action force V_d),
– most importantly, by resisting the external shear force by an additional force V_s.
8. Narrate the advantages and disadvantages of inclined stirrups compared to vertical stirrups.
   - Inclined stirrups are more efficient, acting perpendicular to diagonal tension crack, allowing wider spacing.
   - Are more difficult to construct and can be vulnerable/ineffective for reversible loading.
9. Narrate the ACI code provisions for the flexural shear design of beams by WSD and USD. For both methods, mention
   - V_{ext} and V_n.
   - V_c, V_{c1} and V_{c2}.
   - Allowable maximum and minimum spacing.
10. Explain the effect of axial force on the shear strength of concrete.
    - Qualitatively explain how normal force (compression/tension) effects frictional (shear) resistance.
    - Mention the formulae with graph showing v_c vs. N/A_g.
11. Mention the distinctive features of the shear design of deep beams. For deep and normal beams, mention
    - v_c, v_n, and A_{vh}
12. What is one-way slab? Give some common examples of one-way slabs in engineering structures.
    - Slab where loads are transferred in one direction only.
    - Obviously if supports are in one direction, but span ratios and support stiffness may transfer much of the load in one direction, approximating one-way action.
    - Examples include slab bridge/culvert, garage, balcony, porch, staircase, etc.

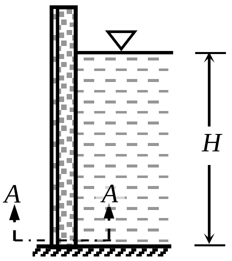
13. Narrate the ACI code provisions for choosing the minimum thickness of one-way slabs. Explain why the required thickness of one-way slabs increase with the yield strength of reinforcing steel.
    - Mention the provisions (considering deflections) depending on support conditions.
    - Since E_s is almost constant, stronger steels have larger yield strains and deflections, requiring thicker slabs.
14. Narrate the necessity and ACI code provisions for temperature and shrinkage reinforcement in slabs. Explain why the minimum flexural steel required by beams is not applicable for slabs.
    - Temperature drop and loss of water results in concrete volume reduction. Adjoining structural elements and supports tend to check the reduction, creating tension in concrete and crack formation.
    - Temperature and shrinkage reinforcements tend to check the cracks, keeping them to hairline cracks.
    - Mention the formulae (ACI for stone, BNBC for brick aggregate) and maximum spacing.
    - Minimum flexural steel for beams is not applicable for slabs, which are two-dimensional, much larger in size and often statically indeterminate.
15. Explain why shear reinforcements are usually not provided in the design of RC slabs.
    - Slab thickness based on serviceability (deflections) requirements is often large enough for flexural shear.
    - Shear reinforcements are difficult to place in slabs.
16. What is development length? Explain the factors influencing development length of deformed bars in tension.
    - The length of embedment necessary to develop the full tensile strength of the bar.
    - Factors include
      - Bar diameter and Yield strength of steel: Larger bars and stronger steel require larger tensile forces in steel, which requires larger l_d to be transferred to concrete.
        Moreover, smaller bars require less l_d than predicted by empirical equations developed for larger bars.
      - Tensile strength of concrete: The most common type of bond failure is splitting of concrete, associated with tension failure of concrete. Therefore, l_d is inversely proportional to tensile strength of concrete.
      - Cover distance and Bar spacing: If concrete clear cover or bar spacing is increased, more concrete is available to resist the tension, decreasing the development length.
      - Lateral reinforcement: Provided by stirrups, they tend to check the opening and propagation of cracks. Their effectiveness depends on the strength, area and spacing along the development length.
      - Bar location: Bars on substantial concrete require more l_d due to tendency for excess water and entrapped air to rise to the top during vibration.
      - Bar coating: Often used to prevent corrosion of reinforcing bars, they reduce the bond-strength by preventing the adhesion between concrete and steel. Development length of such bars increase as a result.
      - Type of concrete: Since tensile strength of lightweight concrete is less than for normal-density concrete, development lengths must be increased for them
17. Explain why the development length of compression bars is smaller than that of tension bars.
    - Bars subjected to compression transfer loads in the form of bond along its length as well as end bearing.
    - Bond is also better along bar length, because concrete is less cracked than in tension.
18. Narrate the necessity of hook and anchorage for plain and deformed bars.
    - Hooks prevent bond failures by providing end anchorage of bars with concrete.
    - Provide almost the entire bond for plain bars, through 'tied-arch action'.
19. What are bar splices? Distinguish between lap splices in tension and compression.
    - Since it is often more convenient to work with shorter bar lengths, bars are often spliced on site. Splicing of bars at maximum stress should be avoided, and they should be staggered when used.
    - Two different types of lap splices (Class A and B) are used for *bars in tension*, corresponding to minimum length of lap, depending on area of reinforcement compared to area required by analysis.
    - Minimum lap length for compression splices depends on steel strength.
    - Mention formulae for both tension and compression splices.
20. Explain why flexural reinforcement bars are not cut off exactly where they are not theoretically required.
    Also justify the difference among the approximate bar cut-off points (at L_n/3, L_n/4, L_n/7, L_n/8 from supports).
    - Tensile steel should never be discontinued exactly at the theoretically described points, because of
      - Propagation of diagonal tension cracks.
      - Difference between design and actual moment diagrams due to approximations of loads and analysis.
    - For uniformly distributed beams of nearly equal spans
      - L_n/3 approximates negative bar cut-off point for continuous ends (large moments) and L_n/4 for discontinuous ends (less or almost no moment)
      - L_n/4 approximates crank locations for continuous ends and L_n/7 for discontinuous ends, which are smaller than cut-off distances as no extension or l_d is involved.
      - L_n/8 approximates the point where about 50% positive bars can be cut-off, considering extension.

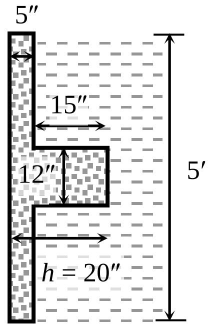
Final Examination Spring 2011 (Set 1)

Given: $f_c' = 3$ ksi, $f_y = 50$ ksi
$\therefore n = 9$, $f_{call} = 0.45 f_c' = 1.35$ ksi, $f_{sall} = 0.4 f_y = 20$ ksi, $r = f_{sall}/f_{call} = 14.81$
$k = n/(n + r) = 0.378$, $j = 1 - k/3 = 0.874$, $R = 0.5 f_{call}\, k\, j = 0.223$ ksi

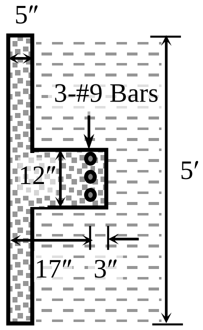
Also $\rho_b = (\alpha f_c'/ f_y)\,(87/(87 + f_y) = 0.0274$, $\rho_{max} = 0.75\, \rho_b = 0.0206$
Using $\phi = 0.90$, $R_u = \phi\, \rho_{max}\, f_y\, (1 - 0.59\, \rho_{max}\, f_y/f_c') = 0.739$ ksi

5. The side elevation and cross-section *A*-*A* of the wall-beam of a water tank is shown in the figure below. Use WSD to calculate the maximum allowable height (H) of water in the tank if the section is
(i) made of plain concrete, (ii) reinforced with 3-#9 bars (as shown).





(i) Plain Concrete Section *A*-*A*



(ii) Reinforced Concrete Section *A*-*A*

Solution

For water height = H, width $B = 5$ ft, and unit weight $\gamma_{water} = 0.0625$ k/ft^3
Maximum bending moment, $M_{max} = \gamma_{water} BH^3/6 = 0.0521H^3$ k-ft

(i) $\bar{x} = (300 \times 2.5 + 180 \times 12.5)/480 = 6.25''$
$\bar{I} = 60 \times 5^3/12 + 300 \times 3.75^2 + 12 \times 15^3/12 + 180 \times 6.25^2 = 15250$ in^4

Tensile strength of concrete $f_t = 6\sqrt{f_c'} = 0.329$ ksi
$\therefore f_t = M_{max}\, \bar{x}/\bar{I} \Rightarrow 0.329 = 0.0521H^3 \times 12 \times (20 - 6.25)/15250 \Rightarrow H = 8.35$ ft

(ii) $A_s = 3 \times 1.0 = 3.0$ in^2 $\Rightarrow n\rho_s = 9 \times 3.0/(60 \times 17) = 0.0265$
$\Rightarrow k = -n\rho_s + \sqrt{\{2n\rho_s + (n\rho_s)^2\}} = 0.205$, $kd = 3.49'' < t \Rightarrow$ Rectangular Beam
$\therefore j = 1 - k/3 = 0.932$
$\Rightarrow M_c = 0.5 f_{call}\, kj\, B\, d^2 = 186.4$ k-ft and $M_s = A_s f_{sall}\, jd = 950.3$ k-in $= 79.2$ k-ft

$\therefore 0.0521H^3 = 79.2 \Rightarrow H = 11.50$ ft

6. In the water tank shown in Question 5, neglect the effect of the wall and use the WSD method to
(i) calculate the maximum allowable height (H) of water as well as the beam reinforcement (A_s), assuming the section *A*-*A* to be a singly reinforced rectangular beam [with $b = 12''$, $d = 17''$],
(ii) design the section *A*-*A* for the same water height (H) calculated in (i), but with $d = 14''$.

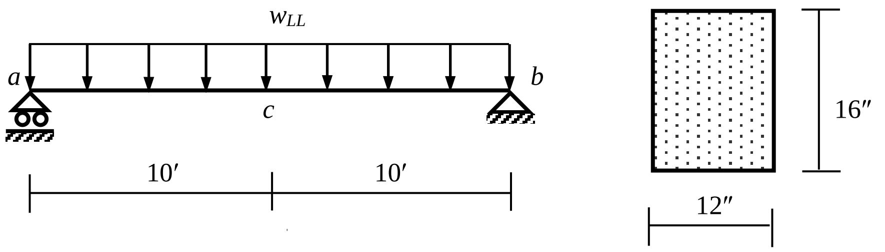
Solution

(i) $d = 17'' \Rightarrow M_c = Rbd^2 = 64.44$ k-ft $\Rightarrow 0.0521H^3 = 64.44 \Rightarrow H = 10.74$ ft
and $A_s = M_s/(f_{sall}\, jd) = 2.60$ in^2; i.e. , 2#7, 2#8 bars

(ii) $d = 14'' \Rightarrow M_c = M_1 = Rbd^2 = 43.70$ k-ft $\Rightarrow M_2 = M_{max} - M_1 = 20.74$ k-ft
$\therefore d' = 3'' \Rightarrow A_s = M_1/(f_{sall}\, jd) + M_2/\{f_{sall}\,(d - d')\} = 3.27$ in^2; i.e., 2#7, 2#9 bars

Also $f_s' = 2\, f_{sall}\, (k - d'/d)/(1 - k) = 10.52$ ksi, which is $< f_{sall}$
$\Rightarrow A_s' = M_2/\{f_s'(d - d')\} = 2.15$ in^2; i.e., 2#7, 1#9 bars

7. Use USD to design the simply supported beam *acb* loaded as shown below (with neat sketches of section *c*), if working live load w_{LL} is (i) zero, (ii) 1 k/ft, (iii) 2 k/ft
[Note: Include beam self-weight and check steel ratio in each case].



Solution

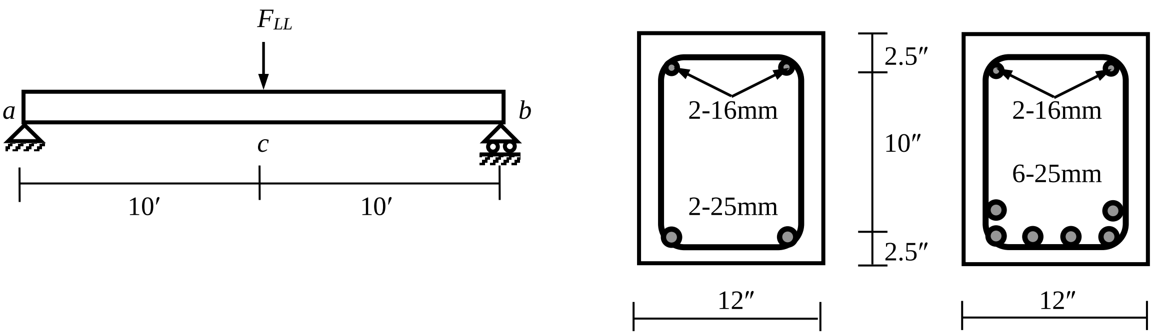
Self-weight $w_{SW} = 12 \times 16/12^2 \times 0.15 = 0.20$ k/ft and span $L = 20$ ft

(i) $w_{LL} = 0 \Rightarrow w_u = 1.4 \times 0.20 = 0.28$ k/ft $\Rightarrow M_u = w_u L^2/8 = 14$ k-ft, $M_n = M_u/\phi = 15.56$ k-ft
One layer of steel $\Rightarrow d = 16 - 2.5 = 13.5''$, and $b = 12'' = 1'$
$\Rightarrow a = d\,[1 - \sqrt{\{1 - 2M_n/(0.85 f_c' bd^2)\}}] = 0.46'' \Rightarrow A_s = 0.85 f_c' ab/f_y = 0.28$ in^2
$\therefore \rho_s = A_s/bd = 0.28/(12 \times 13.5) = 0.0017$, while $\rho_{min} = 6\sqrt{f_c'}/f_y = 0.0033$
$\Rightarrow A_s = \rho_{min}\, bd = 0.53$ in^2; i.e., 2#5 bars

(ii) $w_{LL} = 1$ k/ft $\Rightarrow w_u = 1.4 \times 0.20 + 1.7 \times 1.0 = 1.98$ k/ft $\Rightarrow M_u = w_u L^2/8 = 99$ k-ft, $M_n = M_u/\phi = 110$ k-ft
One layer of steel $\Rightarrow d = 16 - 2.5 = 13.5''$, and $b = 12'' = 1'$
$\Rightarrow a = d\,[1 - \sqrt{\{1 - 2M_n/(0.85 f_c' bd^2)\}}] = 3.70'' \Rightarrow A_s = 0.85 f_c' ab/f_y = 2.27$ in^2
$\therefore \rho_s = A_s/bd = 2.27/(12 \times 13.5) = 0.0140 > \rho_{min}$ and $< \rho_{max} \Rightarrow A_s = 2.27$ in^2; i.e., 3#8 bars

(iii) $w_{LL} = 2$ k/ft $\Rightarrow w_u = 1.4 \times 0.20 + 1.7 \times 2.0 = 3.68$ k/ft $\Rightarrow M_u = w_u L^2/8 = 184$ k-ft, $M_n = M_u/\phi = 204.4$ k-ft
Two layers of steel $\Rightarrow d = 16 - 4 = 12''$, and $b = 12'' = 1'$, $M_{max} = R_u bd^2 = 0.739 \times 12 \times 12^2/12 = 106.35$ k-ft
$\Rightarrow a = d\,[1 - \sqrt{\{1 - 2(M_{max}/\phi)(0.85 f_c' bd^2)\}}] = 4.84'' \Rightarrow A_{s1} = 0.85 f_c' ab/f_y = 2.96$ in^2
$\therefore M_2 = 184 - 106.35 = 77.65$ k-ft $\Rightarrow A_{s2} = M_2/\phi f_y\,(d - d') = 77.65 \times 12/(0.90 \times 50 \times (12 - 2.5)) = 2.18$ in^2
$\Rightarrow A_s = A_{s1} + A_{s2} = 5.14$ in^2; i.e., 4#8 and 2#9 bars
$\therefore c = 0.85a/\alpha = 5.71''$, $f_s' = 87\,(c - d')/c = 48.91$ ksi, which is $< f_y$
$\therefore A_s' = M_2/\phi f_s'\,(d - d') = 2.23$ in^2; i.e., 3#8 bars at the top

8. For the beam *acb* shown below, calculate the steel ratios at *a*, *b*, *c* and use the USD to determine the
(i) maximum working live load F_{LL} the beam can sustain,
(ii) distance from the center *c* where only two bars are required as bottom reinforcement
[Note: Neglect the self-weight of the beam and use simplified equation for development length].



Section *a*, *b*

Section *c*

Solution

(i) At Section *c*, $A_s = 6 \times 0.76 = 4.56$ in^2, $\rho_s = A_s/bd = 4.56/(12 \times 12.5) = 0.0304$, which is $> \rho_b$
$\therefore a = (A_s - A_s')\, f_y/(0.85 f_c' b) = 3.94 \times 50/(0.85 \times 3 \times 12) = 6.44'' \Rightarrow c = (0.85/\alpha)\, a = 7.60''$
Actual stress in compression rod, $f_s' = 87\,(c - d')/c = 87\,(7.60 - 2.5)/7.60 = 58.4$ ksi $> f_y \Rightarrow f_s' = 50$ ksi
$M_u = \phi\,(A_s - A_s')\, f_y\,(d - a/2) + \phi A_s' f_y\,(d - d') = 0.90 \times \{3.94 \times 50\,(12.5 - 6.44/2) + 0.62 \times 50\,(12.5 - 2.5)\}/12$
$= 160.38$ k-ft
$\therefore 1.7\, F_{LL}\, L/4 = M_u/\phi \Rightarrow 1.7\, F_{LL} \times 20/4 = 160.38/0.90 = 178.20 \Rightarrow F_{LL} = 20.96$ k

(ii) At Section *a*, $\rho_s = A_s/bd = 1.52/150 = 0.0101 < \rho_b$, and $\rho_s' = A_s'/bd = 0.62/150 = 0.0041$
$\therefore$ Neglecting effect of compression rod, $a = A_s f_y/(0.85 f_c' b) = 1.52 \times 50/(0.85 \times 3 \times 12) = 2.48''$
$M_u = \phi A_s f_y\,(d - a/2) = 0.90 \times \{1.52 \times 50\,(12.5 - 2.48/2)\}/12 = 64.17$ k-ft
This is at a distance *x* from support, where $x = 64.17/160.38 \times 10 = 4$ ft; i.e., 6 ft from center
Simplified equation for $l_d = 0.05(f_y/\sqrt{f_c'})\, d_b = 44.92'' = 3.74$ ft; $\therefore$ Distance from center $= 6 + 3.74 = 9.74$ ft

9. Use WSD to design (and sketch) the beam described in Question 7 for shear, using
   (i) diagonal stirrups for w_{LL} = 1 k/ft [α = 45°], with one layer of longitudinal bars,
   (ii) perpendicular (vertical) stirrups for w_{LL} = 2 k/ft, with two layers of longitudinal bars.

Solution

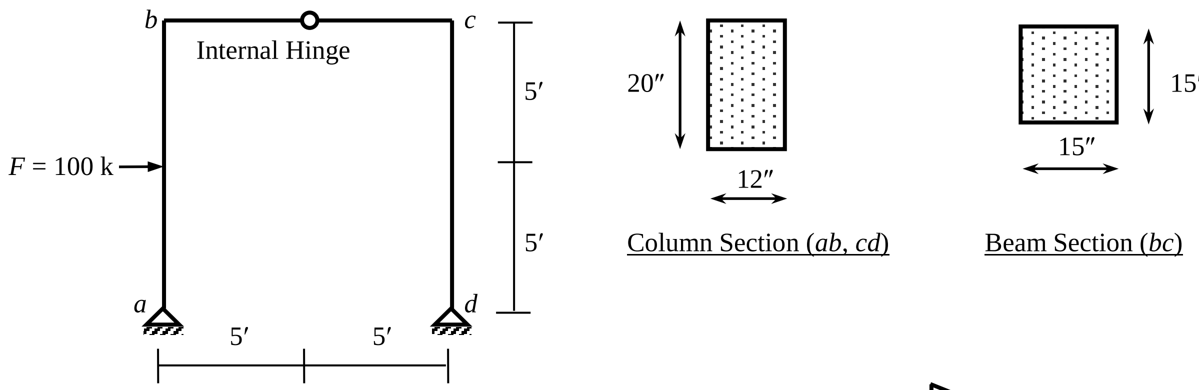
(i) w_{LL} = 1 k/ft ⇒ w = 0.20 + 1.0 = 1.20 k/ft
One layer of steel ⇒ d = 16 – 2.5 = 13.5″, and b = 12″ = 1′
Assuming 12″ columns ⇒ $V = w(L/2–c/2–d)$ = 10.05 k, $V_c = 1.1\sqrt{f_c'}\,bd = 1.1\sqrt{(3/1000)} \times 12 \times 13.5$ = 9.76 k
$S = A_v f_v\, d/(V–V_c)\,(\sin\alpha + \cos\alpha) = 0.22 \times 20 \times 13.5/(10.05–9.76)\,(\sin 45° + \cos 45°)$ = 145.8″

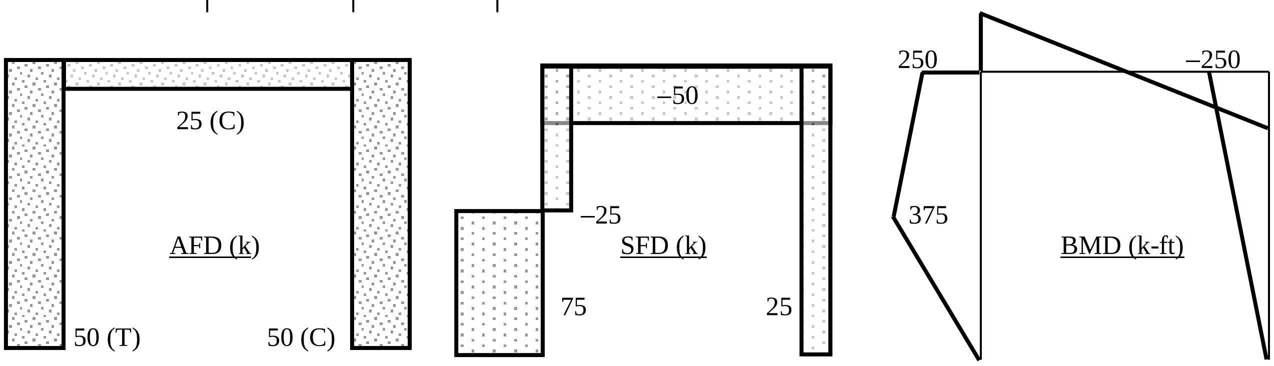
But $S \le$ 12″, d/2 (= 6.75″), $A_v f_y/(50 b_w)$ [= 0.22 × 50000/(50 × 12) = 18.33″] ⇒ Use #3 @6.5″ c/c

(ii) w_{LL} = 2 k/ft ⇒ w = 0.20 + 2.0 = 2.20 k/ft
One layer of steel ⇒ d = 16 – 4 = 12″, and b = 12″ = 1′
Assuming 12″ columns ⇒ $V = w(L/2–c/2–d)$ = 18.70 k, $V_c = 1.1\sqrt{(3/1000)} \times 12 \times 12$ = 8.68 k
$S = A_v f_v\, d/(V–V_c) = 0.22 \times 20 \times 12/(18.70–8.68)$ = 5.41″

But $S \le$ 12″, d/2 (= 6″), $A_v f_y/(50 b_w)$ [=18.33″] ⇒ Use #3@5″c/c

10. Figure below shows AFD, SFD and BMD of frame *abcd* loaded as shown (for working live load *F*).
Use USD to design the frame for shear (considering effect of axial force), using two layers of longitudinal bars and perpendicular stirrups.



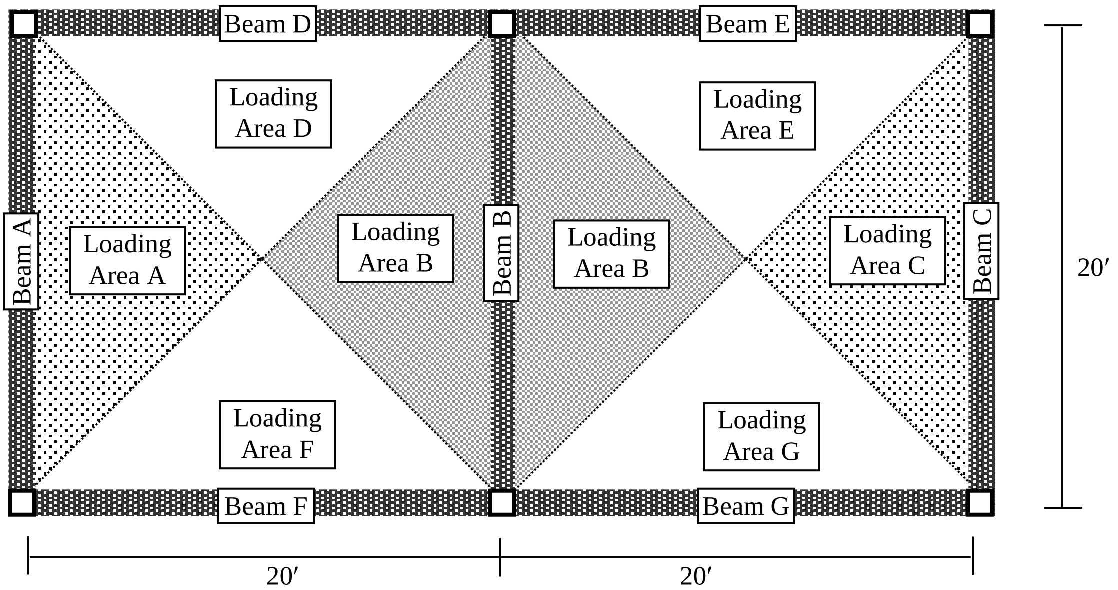


Solution

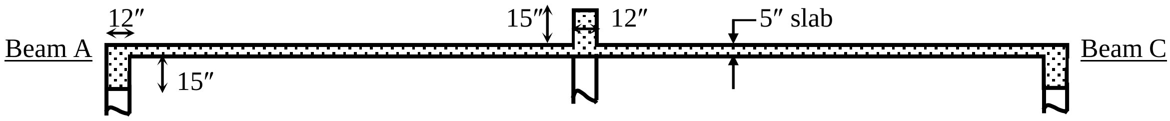
Maximum (factored) shear forces for *ab*, *bc*, *cd* are (75 × 1.7 =) 127.5 k, (50 × 1.7 =) 85 k, (25 × 1.7 =) 42.5 k
Maximum (factored) axial forces for *ab*, *bc*, *cd* are (–50 × 1.7 =) 85 k, (25 × 1.7 =) 42.5 k, (50 × 1.7 =) 85.5 k

	V_u	V_n	N_u	b	h	*Shear Capacity Factor*	d	V_c	S	S_{max}	*Comments*
ab	127.5	150	-85	12	20	0.29	16	6.13	2.22	8.0	Change Section
bc	85.0	100	42.5	15	15	1.09	11	19.78	2.74	5.5	#4 @2.5″c/c
cd	42.5	50	85	12	20	1.18	16	24.76	12.68	8.0	#4 @5.5″c/c

11. The plan view of a slab-beam system is shown in the figure below, where the beams carry loads on the areas denoted. Use USD to design the beams A and B (with neat sketches), if the slabs carry working loads FF = 20 psf, RW = 50 psf and LL = 60 psf, in addition to self-weight.



Beam B



Solution

Slab Load w_{Slab} = 1.4 (5/12 × 150 + 20 + 50) + 1.7 × 60 = 287.5 psf

Load w_A on Beam A = (287.5/1000) × 10/2 + 1.4 × 12 × 15/12^2 × 0.15 = 1.70 k/ft ⇒ $M_{A(max)} = w_A L_A^2/8$ = 85 k-ft

b_{eff} = Min (6t + b_w, L/12 + b_w, S_{cl}/2 + b_w) = Min (30 + 12, 20 × 12/12 +12, 9.5 × 12 + 12) = 32″

$M_f = 0.85 f_c'(b_{eff} - b_w)\, t\, (d - t/2)$ = 2.55 × (32 – 12) × 5 × (17.5–2.5)/12 = 318.8 k-ft, which is > $M_{A(max)}$

∴ Design as Rectangular beam ⇒ $a = d\,[1 - \sqrt{\{1 - 2(M_{max}/\phi)(0.85 f_c' b_{eff} d^2)\}}]$

= 17.5 [1 – √{1 – 2(85 × 12/0.90)(2.55 × 32 × 17.5^2)}] = 0.81″, which is < t

∴ $A_s = 0.85 f_c' a b_{eff} / f_y$ = 1.33 in^2; i.e., 3-#6 bars

Load w_B on Beam A = (287.5/1000) × 10 + 1.4 × 12 × 15/12^2 × 0.15 = 3.14 k/ft ⇒ $M_{B(max)} = w_B L_B^2/8$ = 156.9 k-ft

b_{eff} = Min (16t + b_w, L/4, S_c) = Min (80 + 12, 20 × 12/4, 20 × 12) = 60″

$M_f = 0.85 f_c'(b_{eff} - b_w)\, t\, (d - t/2)$ = 2.55 × (60 – 12) × 5 × (17.5–2.5)/12 = 765 k-ft, which is > $M_{B(max)}$

∴ Design as Rectangular beam ⇒ a = 17.5 [1 – √{1 – 2(156.9 × 12/0.90)(2.55 × 60 × 17.5^2)}] = 0.80″, which is < t

∴ $A_s = 0.85 f_c' a b_{eff} / f_y$ = 2.45 in^2; i.e., 4-#7 bars

12. In the floor system described in Question 11, calculate required slab thickness and reinforcements if
    (i) Beams D, E, F, G are removed (i.e., slabs are supported on Beams A, B, C),
    (ii) Beam B is removed (i.e., slabs are supported on Beams A, C, D, E, F, G),
    (iii) Beams A, C, D, E, F, G and corner columns are removed (i.e., slabs are supported on Beam B).

Solution

Thickness multiplication factor = 0.4 + f_y/100 = 0.90

(i) One Continuous beam with clear span L = 19′ ⇒ t = 19/24 × 12 × (0.90) = 8.55″; i.e., t = 8.5″, d = 7.5″

w = 8.5 × 150/12 + 20 + 50 + 60 = 236.25 psf

$M^{(-)} = wL^2/9$ = 9.48 k-ft/ft, $M^{(+)} = wL^2/14$ = 6.09 k-ft/ft ⇒ $d_{req} = \sqrt{(M_{max}/R)}$ = 6.52″, which is < 7.5″

$A_s^{(-)} = M^{(-)}/(f_{s(all)}\, j\, d)$ = 0.87 in^2/ft, $A_s^{(+)} = M^{(+)}/(f_{s(all)}\, j\, d)$ = 0.56 in^2/ft, A_{st} = 0.03 t = 0.26 in^2/ft

(ii) Simply supported beam with clear span L = 19′ ⇒ t = 19/20 × 12 × (0.90) = 10.26″; i.e., t = 10.5″, d = 9.5″

w = 10.5 × 150/12 + 20 + 50 + 60 = 261.25 psf

$M^{(+)} = wL^2/8$ = 11.79 k-ft/ft ⇒ $d_{req} = \sqrt{(M_{max}/R)}$ = 7.27″, which is < 9.5″

$A_s^{(+)} = M^{(+)}/(f_{s(all)}\, j\, d)$ = 11.79 × 12/(20 × 0.874 × 9.5) = 0.85 in^2/ft, A_{st} = 0.03 t = 0.32 in^2/ft

(iii) Cantilever beam with clear span L = 19.5′ ⇒ t = 19.5/10 × 12 × (0.90) = 21.06″; i.e., t = 21″, d = 20″

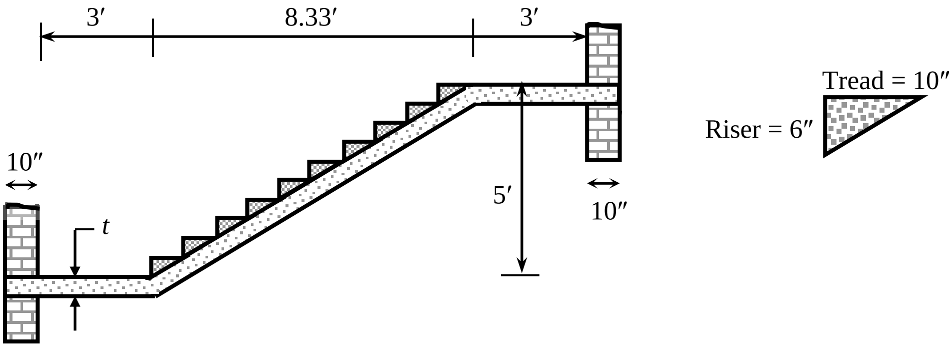
w = 21 × 150/12 + 20 + 50 + 60 = 392.5 psf

$M^{(-)} = wL^2/2$ = 74.62 k-ft/ft ⇒ $d_{req} = \sqrt{(M_{max}/R)}$ = 18.29″, which is < 20″

$A_s^{(-)} = M^{(-)}/(f_{s(all)}\, j\, d)$ = 74.62 × 12/(20 × 0.874 × 20) = 2.56 in^2/ft, A_{st} = 0.03 t = 0.63 in^2/ft

13. Figure below shows a staircase simply-supported on 10″ brickwalls.

Determine the thickness (t) of the waist-slab and use the WSD to calculate the
(i) allowable live load on the staircase if FF = 20 psf,
(ii) required reinforcements in the slab [and show them with neat sketch].



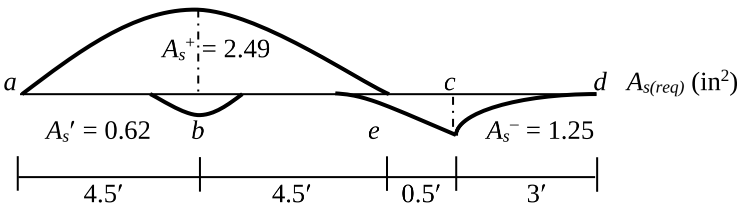
Solution

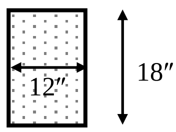
Thickness multiplication factor = 0.4 + f_y/100 = 0.90

Simply supported beam with clear span L = 14.33′ ⇒ t = 14.33/20 × 12 × (0.90) = 7.74″; i.e., t = 8″, d = 7″
$M_{all}^{(+)} = Rbd^2$ = 0.223 × 1 × 7^2 = 10.92 k-ft/ft
$M_{all}^{(+)} = w_{all}L^2/8$ = 10.92 ⇒ w_{all} = 425.62 psf ⇒ $w_{LL(all)}$ = 425.62 – (8 + 3) × 150/12 – 20 = 268.12 psf

$A_s^{(+)} = M^{(+)}/(f_{s(all)}j\ d)$ = 10.92 × 12/(20 × 0.874 × 7) = 1.07 in^2/ft; i.e., #5 @3.5″c/c, OR #6 @5″c/c
A_{st} = 0.03t = 0.24 in^2/ft; i.e., #3 @5.5″c/c

14. Figure below shows the required longitudinal steel area and cross-section of a RC beam (normal-weight concrete, no epoxy coating) supported by 12″-dia columns at a and c. If #3 stirrups are spaced @5″c/c around a, c and @8″ around b, calculate the
<u>allowable longitudinal bar diameter</u>, <u>required number of bars</u> and <u>lap length (with suitable location)</u> for A_s^+, A_s' and A_s^-, (i) with end anchorage, (ii) without end anchorage.





Beam Section

Bars must extend l_d beyond where they are not required, allowable length = 0.5 + 3 = 3.5′ = 42″

Normal-weight concrete ⇒ λ = 1.0, no epoxy coating ⇒ β = 1.0
Also α = 1.0 for bottom bars and = 1.3 for top bars
while γ = 1.0 for larger (bottom) bars and 0.8 for smaller (top) bars

(i) Using $l_d \cong 0.05\ (f_y/\sqrt{f_c'})\ (\alpha\beta\gamma\lambda)\ d_b$ = 0.05 (50000/√30000) (1.0) d_b = 45.64 d_b = 42″ ⇒ d_b = 0.92″

Assuming 2 #8, 2 #6 bars at bottom, with c = Smallest bar spacing or cover dimension = 1.5″
and K_{tr} = Transverse Reinforcement Index = $A_{tr}\ f_{tr}/(1.5Sn)$ = 0.22 × 50/(1.5 × 8 × 4) = 0.23″
∴ $(c + K_{tr})/d_b$ = (1.5 + 0.23)/1 = 1.73

For #8 bars, using $l_d = (3/40)\ (f_y/\sqrt{f_c'})\ (\alpha\beta\gamma\lambda)/\{(c + K_{tr})/d_b\}d_b$
= (3/40) (50000/√3000) (1.0)/{1.73}(1″) = 39.6″< 42″, OK; ∴Use 2#8, 2 #6 A_s^+ bars

Larger extensions possible for A_s' and A_s^-, but less bars required ⇒ A_s' = 2 #5, and A_s^- = 4 #5 will be OK

Class B bars ⇒ Lap Length = 1.3l_d = 51.5″ (maximum) for A_s^+ and A_s^-, and = 0.5$f_y d_b$ = 25″ (maximum) for A_s' and can be provided around e

(ii) Using $l_d \cong 0.02\ (f_y/\sqrt{f_c'})\ (\alpha\beta\gamma\lambda)\ d_b$ = 0.02 (50000/√30000) (1.0) d_b = 18.26 d_b = 42″ ⇒ d_b = 2.3″
∴For practical purpose, maximum 2 #10 A_s^+ bars can be provided, with l_d = 18.26d_b = 22.83″, and

Lap Length = 1.3l_d = 29.7″
while bars of (i) can be repeated for A_s' and A_s^-