HYDROLOGY - TUTORIAL 4

UNSTEADY FLOW IN CHANNELS

In this tutorial you will

- Derive equations associated with a rise in the level of the bed.
- Define a hydraulic jump and derive the equations for it.
- Define a narrow weir and derive equations for it.
- Define a broad weir and derive equations for it.
- Derive equations for the flow rate through a Venturi Flume.
- Solve questions from past papers.

This tutorial is a continuation of tutorial 1 and 2 and these should be studied first.

UNSTEADY FLOW

If the depth of the water is not constant, we have unsteady flow. This might occur when the frictional losses do not match the change in potential energy. In this case the hydraulic gradient 'i' is not the same as the slope 'S'. The height of the bed relative to the datum level is z.

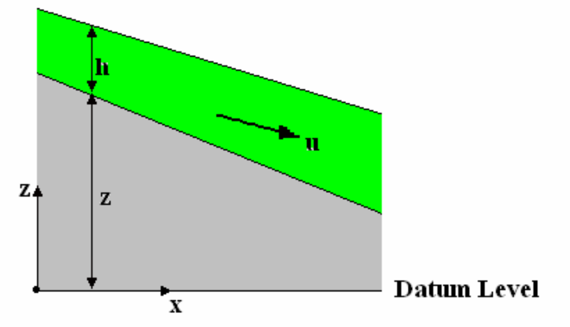


Figure 1

The total head is defined as h_s plus the additional potential head z

$$h_T = h_S + z = h + z + u^2/2g$$

Start with $h_S = h_T - z$

Differentiate with respect to x (the distance along the channel from a given datum point).

$$dh_S /dx = dh_T/dx - dz/dx$$

dz/dx is the gradient of the bed and this is clearly negative so dz/dx = - S

$$dh_S /dx = dh_T/dx + S$$

The change in the total head can only be due to frictional losses and this will be a reduction so we can define this as the hydraulic gradient $i = dh_f/dx = - dh_T/dx$

$$dh_S /dx = S - i$$

We had $h_S = h + u^2/2g$

Differentiate this with respect to h $dh_S/dh = 1 + (u/g)du/dh$

u = Q/A.

Differentiate with respect to A $du/dA = -Q/A^2$

If A is a function of depth, then this is difficult. For a rectangular channel A = B h

Differentiate with respect to h $dA/dh = B$

Substitute $du/dh = (-Q/A^2) B$

Substitute into $dh_S/dh = 1 + (u/g)du/dh$

$$dh_S/dh = 1 - (u/g) (Q/A^2) B$$

$$dh_S/dh = 1 - (u/g) (Q/A^2) B$$

Substitute Q/A = u $dh_S/dh = 1 - (u^2/g) B/A$

Substitute B/A = 1/h $dh_S/dh = 1 - (u^2/gh) = 1 – F_r^2$

For maximum or minimum specific head $dh_S/dh = 0$ and this can only occur if $F_r = 1$

Flow is at the critical depth when $\mathbf{F_r = 1}$

The velocity that produces the critical depth is $\mathbf{u_c = \sqrt{(g\ h)}}$

Note that it has been assumed that h is constant at all widths so the Froude number is only 1 when the channel is rectangular in section.

HYDRAULIC JUMP

It has already been shown that for a given flow rate in an open channel, there are two possible depths. One is when the flow is slow and deep called TRANQUIL FLOW and the other when it is shallow and fast called RAPID FLOW or SHOOTING FLOW.

It is possible rapid flow to change to tranquil flow quite suddenly and spontaneously and when it does we get a phenomenon called a hydraulic jump. It is not possible for the reverse to happen.

For a hydraulic jump to occur, the $F_r > 1$, i.e. the flow must be supercritical.

The jump might occur because the slope of the bed is insufficient for friction to balance the loss of potential energy. Since the losses are smaller for the tranquil flow, the balance can be restored.

A jump can be made to occur if there is an obstacle on the bed higher than the critical depth.

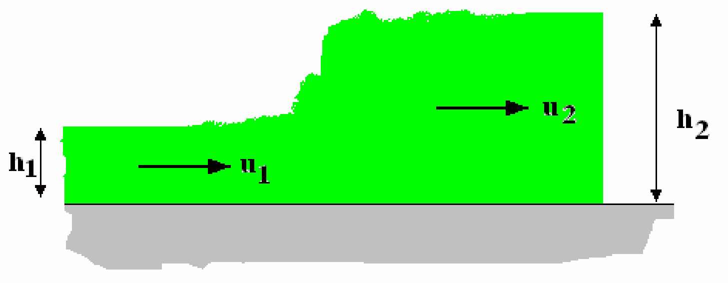


Figure 2

When the change occurs there is a reduction in momentum and an increase in the hydrostatic force. The solution is based on equating them.

u_o = mean velocity. The mean depth is h/2

Pressure force on a cross section is $F_p = \rho gAh/2$ Momentum force at a section $=F_m = \rho Au^2$

The cross sectional area is A = B h where B is the width.

$$\text{Change in pressure force} = \frac{\rho gA_1h_1}{2} - \frac{\rho gA_2h_2}{2} = \frac{\rho gBh_1^2}{2} - \frac{\rho gB_2h_2^2}{2}$$

$$\text{Change in pressure force} = \frac{\rho gB}{2}\left(h_1^2 - h_2^2\right) = \frac{\rho gB}{2}(h_1 - h_2)(h_1 + h_2)$$

Change in momentum force $= \rho Bh_1u_1^2 - \rho Bh_2u_2^2 = \rho B(h_1u_1^2 - h_2u_2^2)$

For continuity of flow $u_2^2 = (u_1 h_1/ h_2)^2$

$$\text{Change in momentum force} = \rho B\left(h_1u_1^2 - \frac{h_2h_1^2}{h_2^2}u_1^2\right) = \rho Bu_1^2\frac{h_1}{h_2}(h_2 - h_1)$$

The change in pressure and momentum forces may be equated.

$$\frac{\rho gB}{2}(h_1 - h_2)(h_1 + h_2) = \rho Bu_1^2\frac{h_1}{h_2}(h_2 - h_1)$$

$$\frac{g}{2}(h_1 + h_2) = u_1^2\frac{h_1}{h_2}$$

$$u_1^2 = (h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2}\ \ldots\ldots\ldots\ldots\ldots(1)$$

In terms of the flow rate

$$Q^2 = B^2h_1^2(h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2} = B^2h_1h_2(h_1 + h_2)\frac{g}{2}$$

The flow per unit width is usually given as $q^2 = gh_1h_2\dfrac{(h_1 + h_2)}{2}$ ....(2)

From (1) $\frac{2h_1u_1^2}{g} = \left(h_1h_2 + h_2^2\right)$ $\quad h_2^2 + h_1h_2 - h_1\frac{2u_1^2}{g} = 0$

h_2 may be solved with the quadratic equation giving:

$2h_2 = -h_1 \pm h_1\sqrt{\left\{1+\frac{8u_1^2}{gh_1}\right\}}$ and since h_2 cannot be negative $2h_2 = -h_1 + h_1\sqrt{\left\{1+\frac{8u_1^2}{gh_1}\right\}}$

Substitute the Froude Number $F_r^2 = \frac{u_1^2}{gh_1}$ $\quad 2h_2 = -h_1 + h_1\sqrt{\left\{1+8F_r^2\right\}}$

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}} - 1\right]$$

Next consider the energy balance before and after the jump.

Energy Head before the jump = $h_1 + u_1^2/2g$ Energy Head after the jump = $h_2 + u_2^2/2g$

Head loss = $h_L = h_1 + u_1^2/2g - h_2 - u_2^2/2g$ $\qquad h_L = h_1 - h_2 + (u_1^2 - u_2^2)/2g$

Continuity of flow $Q = u_1 B h_1 = u_2 B h_2$ $\qquad u_2 = u_1\frac{h_1}{h_2}$

Hence $u_1^2 - u_2^2 = u_1^2 - u_1^2\left(\frac{h_1}{h_2}\right)^2 = u_1^2\left\{1-\left(\frac{h_1}{h_2}\right)^2\right\}$

We already found equation (1) was $u_1^2 = (h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2}$

Substitute

$$u_1^2 - u_2^2 = (h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2}\left\{1-\left(\frac{h_1}{h_2}\right)^2\right\}$$

$$u_1^2 - u_2^2 = \frac{g}{2}(h_1 + h_2)\left(\frac{h_2}{h_1}\right)\left\{\frac{h_2^2 - h_1^2}{h_2^2}\right\}$$

$$\left(u_1^2 - u_2^2\right) = \frac{g\left\{(h_1 + h_2)\left(h_2^2 - h_1^2\right)\right\}}{2h_1h_2}$$

Substitute into the formula for h_L

$h_L = h_1 - h_2 + (u_1^2 - u_2^2)/2g$

$$h_L = (h_1 - h_2) + \frac{g(h_1 + h_2)\left(h_2^2 - h_1^2\right)}{2 \text{ x } 2gh_1h_2} = (h_1 - h_2) + \frac{(h_1 + h_2)\left(h_2^2 - h_1^2\right)}{4h_1h_2}$$

$$h_L = \frac{4h_1h_2(h_1 - h_2) + \left\{(h_1 + h_2)\left(h_2^2 - h_1^2\right)\right\}}{4h_1h_2}$$

$$h_L = \frac{4h_1^2h_2 - 4h_1h_2^2 + h_2^3 - h_1^3 + h_1h_2^2 - h_1^2h_2}{4h_1h_2}$$

$$h_L = \frac{3h_1^2h_2 - 3h_1h_2^2 + h_2^3 - h_1^3}{4h_1h_2}$$

$$h_L = \frac{(h_2 - h_1)^3}{4h_1h_2}$$

Useful website http://www.lmnoeng.com/Channels/HydraulicJump.htm

WORKED EXAMPLE No.1

50 m^3/s of water flows in a rectangular channel 8 m wide with a depth of 0.5 m. Show that a hydraulic jump is likely to occur.

Calculate the depth after the jump and the energy loss per second.

SOLUTION

$A_1 = 8 \times 0.5 = 4\ m^2$ $u_1 = Q/A_1 = 50/4 = 12.5$ m/s
Froude Number $F_{r1} = u/\sqrt{gh} = = 12.5/\sqrt{(9.81 \times 0.5)} = 31.855$ This is supercritical so a jump is possible.

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}}-1\right] = \frac{0.5}{2}\sqrt{1+8 \times 31.855^2} = 3.748\ m$$

$A_2 = 8 \times 3.748 = 30\ m^2$ $u_2 = Q/A_2 = 50/30 = 1.667$ m/s

$$h_L = \frac{(h_2 - h_1)^3}{4h_1h_2} = \frac{(3.748 - 0.5)^3}{4 \times 0.5 \times 3.748} = 4.573\ m$$

Energy loss = mgh_L = 50 000 x 9.81 x 4.573 = 2.243 MJ/s
m = 50 000 kg/s

SELF ASSESSMENT EXERCISE No. 1

1. Show by applying Newton's Laws that when a hydraulic jump occurs in a rectangular channel the depth after the jump is

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}}-1\right]$$

Go on to show that the head loss is $$h_L = \frac{(h_2 - h_1)^3}{4h_1h_2}$$

2. 40 m^3/s of water flows in a rectangular channel 10 m wide with a depth of 1.0 m. Show that a hydraulic jump is likely to occur. Calculate the depth after the jump and the energy loss per second.
(Answers 1.374 m and 3.735 kJ/s)

3. Water has a depth H = 1.5 m behind a sluice gate and emerges from the gate with a depth of 0.4 m. Downstream a hydraulic jump occurs. Calculate depth after the jump and the mean velocity before and after the jump. (Note use Bernoulli to find u_1)

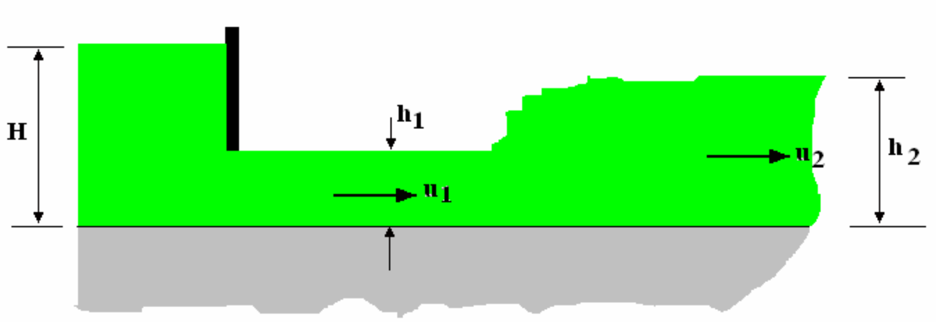


Figure 3

(Answers 1.416 and 1.627 m/s)

RISE IN LEVEL OF BED

In this section we will examine what happens to the level of water flowing in a channel when there is a sudden ride in the level of the bed.

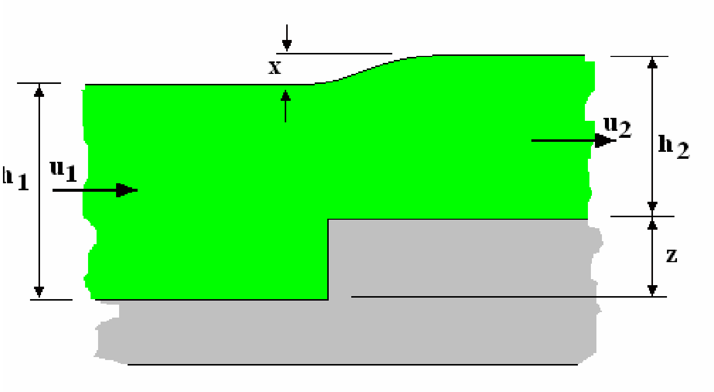


Figure 4

First apply Bernoulli between points (1) and (2)

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} + z$$

From the continuity equation substitute $u_2 = \frac{u_1 h_1}{h_2}$

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{\left(\frac{u_1 h_1}{h_2}\right)^2}{2g} + z$$

Rearrange

$$h_1 + \frac{u_1^2}{2g} - h_2 - \frac{u_1^2 h_1^2}{2g h_2^2} - z = 0$$

$$\left(h_1 + \frac{u_1^2}{2g} - h_2 - z\right)h_2^2 - \frac{u_1^2 h_1^2}{2g} = 0$$

Substitute $h_2 = x + h_1 - z$

$$\left\{h_1 + \frac{u_1^2}{2g} - (x + h_1 - z) - z\right\}(x + h_1 - z)^2 - \frac{u_1^2\{(x + h_1 - z)^2\}}{2g} = 0$$

$$\left\{\frac{u_1^2}{2g} - x\right\}(x + h_1 - z)^2 - \frac{u_1^2\{(x + h_1 - z)^2\}}{2g} = 0$$

$$\left\{u_1^2 - 2gx\right\}(x + h_1 - z)^2 - u_1^2(x + h_1 - z)^2 = 0$$

$$\left\{u_1^2 - 2gx\right\}(x + h_1 - z)^2 - u_1^2(x + h_1 - z)^2 = 0$$

There follows a long bout of more algebra to produce a cubic equation for x :

$$x^3 + 2x^2\left(h_1 - z - \frac{u_1^2}{4g}\right) + x(h_1 - z)\left(h_1 - z - \frac{u_1^2}{g}\right) + z\frac{u_1^2}{2g}(2h_1 - z) = 0$$

This equation may be used to solve the change in height of the water. If the values of x and z are small, we may neglect products and higher powers of small numbers so the equation simplifies to:

$$x\left(h_1 - \frac{u_1^2}{g}\right) + \frac{z u_1^2}{g} = 0$$

$$x\left(1 - \frac{u_1^2}{g h_1}\right) + \frac{z u_1^2}{g h_1} = 0$$

The Froude number approaching the change is $F_{r1} = \frac{u_1}{\sqrt{gh_1}}$ hence

$$x\left(1 - F_{r1}^{\ 2}\right) + zF_{r1}^{\ 2} = 0$$

$$x = \frac{zF_{r1}^{\ 2}}{\left(F_{r1}^{\ 2} - h_1\right)}$$

The equation indicates that if flow is supercritical ($F_{r1}>1$) then x is positive and the surface rises. If the flow is sub critical (tranquil $F_{r1}<1$) then x is negative and the surface is depressed.

WORKED EXAMPLE No. 2

Water flows in a rectangular channel with a depth of 0.55 m and a mean velocity of 4.5 m/s. Downstream there is a rise in the level of the bed of 0.075 m. Determine the depth and mean velocity after the rise. Is Bernoulli's equation is satisfied?

SOLUTION

$z = 0.075$ m $h_1 = 0.55$ m $\quad u_1 = 4.5$ m/s $\quad F_{r1} = \frac{u_1}{\sqrt{gh_1}} = \frac{4.5}{\sqrt{9.81 \text{x} 0.55}} = 1.937$

The flow is supercritical (Rapid) $x = \frac{0.075\,(1.937)^2}{\left(1.937^2 - 0.55\right)} = 0.102\text{m}$

Depth = 0.55 + 0.102 = 0.577 m

$$u_2 = \frac{u_1 h_1}{h_2} = \frac{4.5 \text{ x } 0.55}{0.577} = 4.289\text{m/s}$$

Energy head before rise is $h_1 + \frac{u_1^2}{2g} = 0.55 + \frac{4.5^2}{2g} = 1.582$

Energy Head after the rise is $h_2 + \frac{u_2^2}{2g} + z = 0.577 + \frac{4.289^2}{2g} + 0.102 = 1.616$

There is a small discrepancy.

SELF ASSESSMENT EXERCISE No. 2

1. Water flows in a rectangular channel with a depth of 1.0 m. Downstream there is a rise in the level of the bed of 0.1 m. Determine the mean velocity after the rise and the critical depth upstream if the depth after the rise is :

(a) 1.1 m (Answers 3.13 and 1 m)
(b) 0.8 m (Answer 2.215 m/s and 0.833 m)

WEIRS

We shall consider two forms of weirs, a narrow one with a sharp edge and a broad one with a rounded edge.

NARROW WEIRS

The flow over a sharp edge weir is in essence the same as flow through a rectangular notch. Consider the flow from section (1) to section (2).

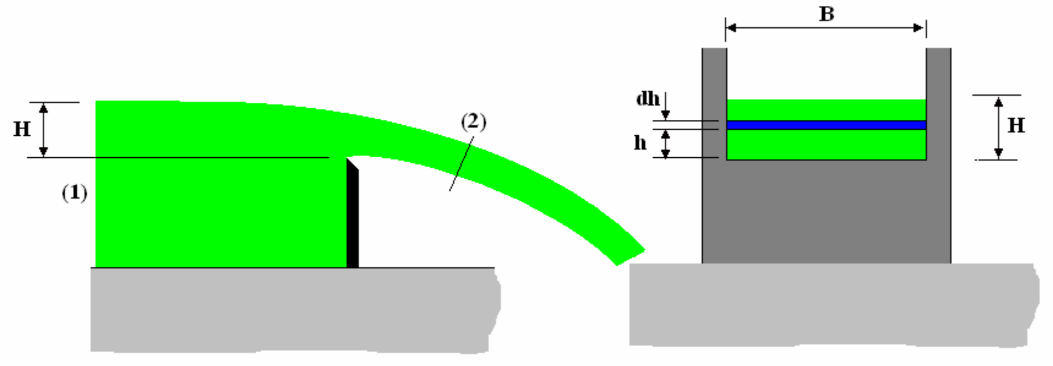


Figure 5

We shall only consider the depth upstream relative to the top of the weir. The mean velocity at a given point is u. The pressure head at (2) is atmospheric Applying Bernoulli between (1) and (2) we have:

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g}$$

The velocity upstream is usually small so we can neglect u_1 and if we use gauge pressure then $h_2 = 0$

$$h_1 = \frac{u_2^2}{2g} \text{ and } u_2 = \sqrt{2gh_1}$$

Next consider a thin horizontal strip at distance h from the bottom of the weir and height dh. The volume flow through it is $dQ = u_2$ Bdh

$$dQ = \sqrt{2gh_1}Bdh$$

$$Q = \sqrt{2g}B\int_0^H h_1^{1/2}dh$$

$$Q = \frac{2}{3}\sqrt{2g}BH^{2/3}$$

It is normal to introduce a coefficient of discharge to correct for losses.

$$Q = \frac{2}{3}C_d\sqrt{2g}BH^{2/3}$$

BROAD CRESTED WEIR

Earlier we examined what happens when the bed of the channel suddenly rises. In the case of the broad crested weir, the level falls as it passes from the weir and it can be proved that at some point on the weir, the flow becomes critical. At this point $h_2 = h_c$.

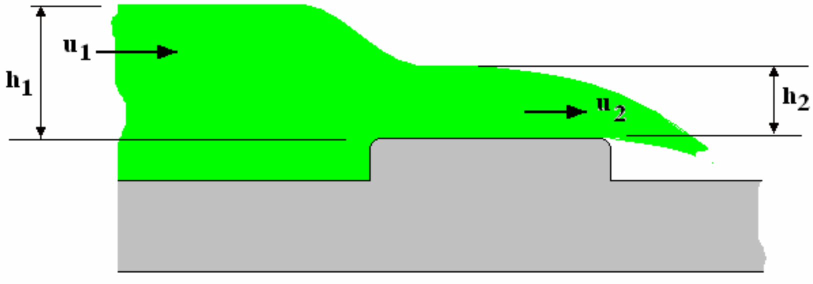


Figure 6

It was shown earlier that $h_c = \frac{2h_s}{3}$ where h_s is the specific total energy head.

It was also shown that the critical flow rate is

$$Q_c = (B + b)g^{1/2}\left(\frac{8h_s^3}{27}\right)^{1/2}$$

For a rectangular section this becomes

$$Q_c = B\left(\frac{8gh_s^3}{27}\right)^{1/2} = 1.705h_s^{3/2}$$

This gives the flow rate over the weir. Usually a coefficient of discharge is used to correct for losses.

$$Q_c = 1.705\, C_d\, h_s^{3/2}$$

If the approach velocity u_1 is negligible then $h_s = h_1$ and makes it easy to solve Q.

WORKED EXAMPLE No. 3

A rectangular channel takes the flow from the foot of a steep spillway with a flow of 10 m^3/s per metre of width. The flow in the channel approaches a broad crested weir with a Froude number of 3. Calculate the following.

i) The mean velocity in the channel.
ii) The minimum height of the weir which will cause a hydraulic jump to occur in the channel.

SOLUTION

The diagram illustrates the problem

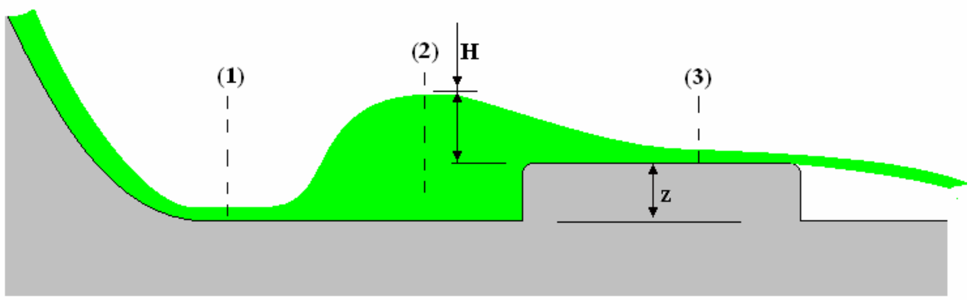


Figure 7

CHANNEL – At section (1) $F_r^2 = 9 = u_1^2/gh_1$ $h_1 = q/u_1$

Combine and $9 = u_1^3/10g$ hence $u_1 = 9.593$ m/s $h_1 = 10/9.593 = 1.042$ m

JUMP

$$h_2 = \frac{h_1}{2}\left[\sqrt{\{1+8F_r^2\}} - 1\right] = 3.932\text{m}$$

WEIR – At section (3) the flow is assumed critical so $F_r = 1$

$$F_r^2 = \frac{u_3^2}{gh_3} = 1 \text{ and } u_3^2 = gh_3$$

$$u_3 = \frac{q}{h_3} \text{ or } h_3 = \frac{q}{u_3} \text{ substitute and } u_3^3 = gq \text{ hence } u_3 = 4.612 \text{ m/s}$$

$h_3 = q/u_3 = 2.168$ m

Bernoulli between (2) and (3)

$$\frac{u_2^2}{2g} + h_2 = \frac{u_3^2}{2g} + h_3 + z \quad \text{hence } z = 1.009$$

$$\text{Check } F_r = \frac{u_3}{\sqrt{gh_3}} = 1$$

SELF ASSESSMENT EXERCISE No. 3

1. A rectangular channel takes the flow from the foot of a spillway with a flow of 20 m^3/s per unit width. The flow in the channel approaches a broad crested weir with a Froude number of 2.24.

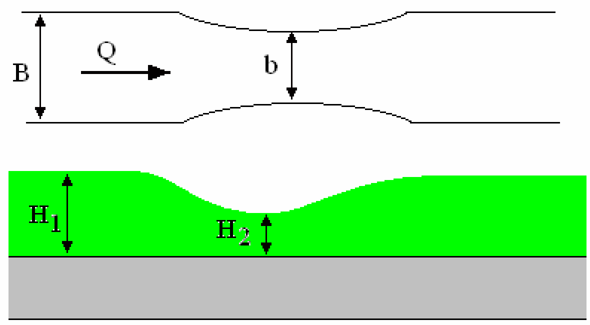
   Calculate the minimum height of the weir to produce a hydraulic jump in the channel.

   (Answer 0.968 m)

2. A wide rectangular spillway has a flow of water of 12 m^3/s per unit width. A broad weir in the path causes a hydraulic jump to occur. The Froude number approaching the jump is 2.5.

   Calculate the minimum height of the weir assuming the flow is critical at some point over it. (Answer 0.85 m)

2. VENTURI FLUME



A venturi flume is a flume that narrows to the throat and then widens back out again. The reduction in width causes a change in velocity and hence height. In a Venturi meter the change is reflected as a change in static pressure but in a flume it is height.

If energy is conserved the total energy head at inlet and at the throat are the same so from Bernoulli we have:

$$H_1 + \frac{u_1^2}{2g} = H_2 + \frac{u_2^2}{2g}$$

The flow rate is $Q = Au = B_1 H_1 u_1 = B_2 H_2 u_2$ Hence $u_1 = u_2 \frac{B_2 H_2}{B_1 H_1}$

$$H_1 + \left(u_2 \frac{B_2 H_2}{B_1 H_1}\right)^2 \frac{1}{2g} = H_2 + \frac{u_2^2}{2g}$$

$$H_1 - H_2 = + \frac{u_2^2}{2g}\left\{1 - \left(\frac{B_2 H_2}{B_1 H_1}\right)^2\right\} \qquad u_2 = \sqrt{\frac{2g(H_1 - H_2)}{1 - \left(\frac{B_2 H_2}{B_1 H_1}\right)^2}}$$

$$Q = B_2 H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{b^2 H_2^2}{B^2 H_1^2}\right]}}$$ Allowing for energy losses $$Q = C_d B_2 H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{b^2 H_2^2}{B^2 H_1^2}\right]}}$$

If the flow rate is a maximum, the depth at the throat will be the critical depth and a hydraulic jump will form downstream of the throat. In this case

$$H_2 = \frac{2}{3} h_s \text{ and } h_s = H_2 + \frac{u_2^2}{2g} = \frac{2h_s}{3} + \frac{u_2^2}{2g} \quad \text{Hence } u_2 = \sqrt{\frac{2gh_s}{3}}$$

$$Q = B_2 H_2 u_2 = B_2 H_2 \sqrt{\frac{2gh_s}{3}} = \frac{2B_2 hs}{3}\sqrt{\frac{2gh_s}{3}}$$

Introducing the coefficient of discharge $Q = 1.705 C_d\ h_s^{2/3}$

This is the same as for Broad Crested weir. Solving Q with this formula is not straight forward because h_s contains the velocity term. The examination often asks for the derivation of this formula.

WORKED EXAMPLE No.4

A rectangular channel is 1.2 m wide and narrows to 0.6 m wide in a venturi flume. The depth at the entrance and throat are 0.6 m and 0.55 m respectively. Calculate the flow rate given Cd = 0.88.

SOLUTION

$$Q = C_d B_2 H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{B_2^2 H_2^2}{B_1^2 H_1^2}\right]}} = 0.88 \text{ x } 0.6 \text{ x } 0.55 \sqrt{\frac{2g(0.05)}{\left[1 - \frac{0.6 \text{ x } 0.55}{1.2 \text{ x } 0.6}\right]^2}} = 0.324 \text{ m}^3/3$$

SELF ASSESSMENT EXERCISE No.4

Show that for critical flow with a Froude Number of 1 in a rectangular channel, the depth of flow yc is related to the specific energy head H by the expression $H = 3y_c/2$

Describe with sketches a broad crested weir and a venturi flume.

Show that for both structures the flow rate is related to the critical depth by the relationship

$$Q = 1.705 C_d H^{3/2}$$

Note the symbols are not the same as in the notes and are the ones used typically in the EC exam.