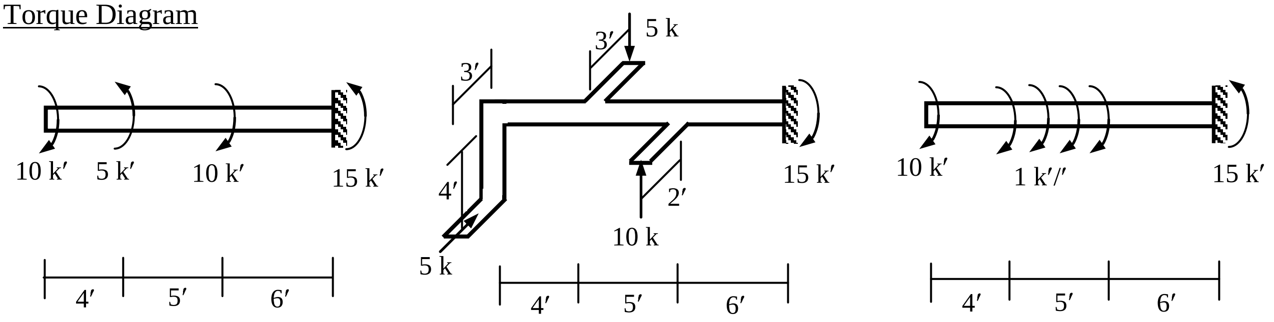
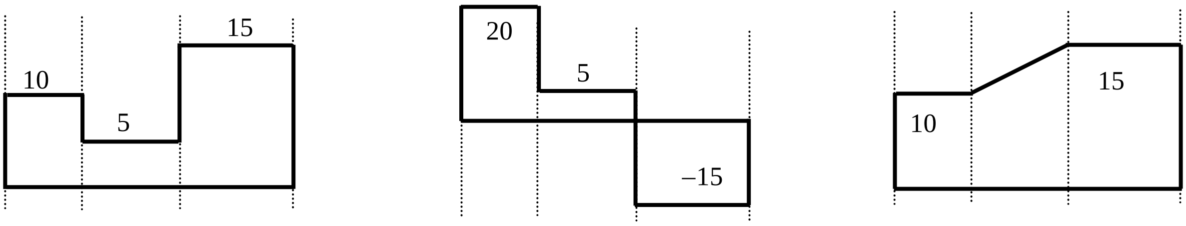
Torque Diagram and Torsional Stress of Circular Section

Torsional or twisting moment is caused by forces whose resultant does not pass through the axis of rotation (called the shear center) of the structural member. Typically, significant torsions are induced in shafts of rotating motors, structural members subjected to eccentric loading (e.g., edge beams) or curved in the horizontal plane (e.g., curved bridges, helical stairs).

Torque Diagram



Case 1: T_1 (k′) Case 2: T_2 (k′) Case 3: T_3 (k′)



Torsion of Circular and Tubular Section

Unlike other sections (e.g., rectangular, thin-walled), the behavior of circular or tubular sections due to torsional moment is well represented by some familiar simplifying assumptions; i.e., plane sections remain plane when torsional moment is applied (i.e., no warping deformations), shear strains are small (as are shear stesses if Hooke's law is valid) and vary linearly from the center of the section.

Using these assumptions, the shear stress at any distance ρ from the center of the circle is $\tau = \tau_{max}\,\rho/c$ ……………………….(i)

$\therefore$ Shear force acting on a small differential area dA is

$dF = \tau\, dA = (\tau_{max}\,\rho/c)\, dA$ ……………………….(ii)

$\therefore$ Corresponding torsional moment, $dT = \rho\, dF = (\tau_{max}\,\rho^2/c)\, dA$ ……..…(iii)

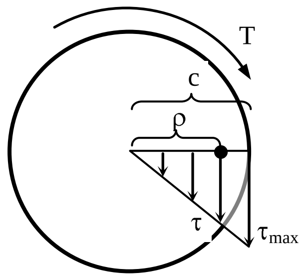
$\therefore$ Total torsional moment integrated over the entire circular area is

$T = \int(\tau_{max}\,\rho^2/c)\, dA = \tau_{max}\, J/c \Rightarrow \tau_{max} = Tc/J$ ………….…………….(iv)

where J = Polar moment of inertia of the section.

For a solid circular section of diameter d, $J = \pi d^4/32$

$\therefore$ Eq. (iv) can be simplified as $\tau_{max} = T(d/2)/(\pi d^4/32) = 16T/\pi d^3$ ………...(v)



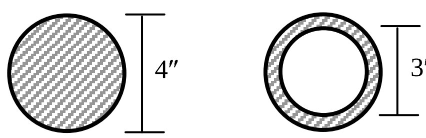
Example: Considering the torque diagram T_1 corresponding to Case 1 shown above, calculate the maximum torsional shear stress for (a) a solid circular section of 4″ diameter, (b) a tubular section of 4″ outside diameter and 3″ inside diameter.

(c) Calculate the required diameter of a solid circular section if the allowable shear stress is 10 ksi.

(a) Since T_{max} = 15 k′ = 180 k″, $\tau_{max} = 16 \times 180/\pi(4)^3$ = 14.32 ksi

(b) $\tau_{max} = 180\times(4/2)/(\pi\{(4)^4-(3)^4\}/32)$ = 20.95 ksi

(c) $\tau_{max} = 16 \times 180/\pi d^3$ = 10 ksi $\Rightarrow$ d = 4.51″



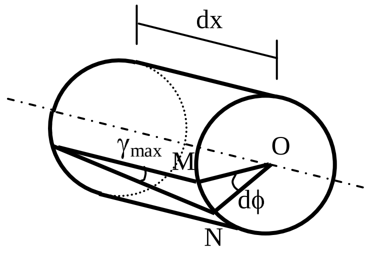
Torsional Rotation of Circular Section

Calculation of torsional rotation is necessary to

1. design structures not only to be strong enough (to withstand torsional stress), but also stiff enough (i.e., they should not deform too much due to torsional moments),
2. design machineries for torsional vibrations,
3. analyze statically indeterminate structures.

Torsional Rotation of Circular and Tubular Section

The assumptions used to derive the equation for torsional shear stress of circular sections are valid here also; i.e., plane sections remain plane due to torsional moment, shear strains (as well as stresses if Hooke's law is valid) are small and vary linearly from the center of the section.



For a cylindrical segment of differential length dx, the length of the arc MN is given by $ds = \gamma_{max}\, dx$ ……………………….(i)

It can also be expressed in terms of the differential angular rotation $d\phi$, i.e.,

$ds = c\, d\phi$ ……………………….(ii)

where c = radius of the circular area.

$\therefore$ Combining Eq. (i) and (ii) $\Rightarrow c\, d\phi = \gamma_{max}\, dx = (\tau_{max}/G)\, dx$

Using $\tau_{max} = Tc/J \Rightarrow c\, d\phi = (Tc/JG)\, dx \Rightarrow d\phi = (T/JG)\, dx$ ………..…(iii)

where J = Polar moment of inertia of the section.

$\therefore$ Integrating between sections A and B $\Rightarrow \phi_B - \phi_A = \int(T/JG)\, dx$ ……….(iv)

where $\int$ is integration between sections A and B.

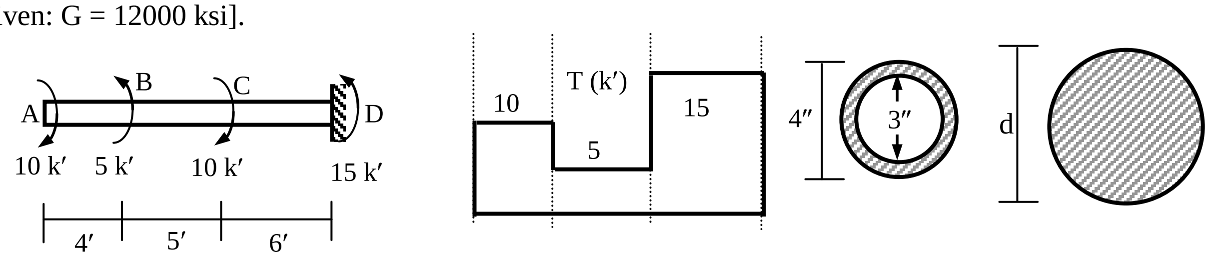
If T, J and G are uniform between A and B, then $\phi_B - \phi_A = (TL/JG)$ …………….(v)

Example: Considering the torque diagram shown below, calculate the torsional rotations for

(a) a tubular section of 4″ outside diameter and 3″ inside diameter.

(b) Calculate the required diameter of a solid circular section if the allowable torsional rotation is 1°

[Given: G = 12000 ksi].



(a) The polar moment of inertia of the tubular section is $J = \pi\{(4)^4-(3)^4\}/32 = 17.18\ in^4$

$\therefore \phi_D - \phi_C = (TL/JG)_{CD} = (15 \times 12) \times (6 \times 12)/(17.18\times12000) = 0.0629$ rad $\Rightarrow \phi_C = -0.0629$ rad $= -3.60°$

$\phi_C - \phi_B = (TL/JG)_{BC} = (5 \times 12) \times (5 \times 12)/(17.18\times12000) = 0.0175$ rad $\Rightarrow \phi_B = -0.0803$ rad $= -4.60°$

$\phi_B - \phi_A = (TL/JG)_{AB} = (10 \times 12) \times (4 \times 12)/(17.18\times12000) = 0.0279$ rad $\Rightarrow \phi_A = -0.1083$ rad $= -6.21°$

(b) Using $\phi_D = 0$, the maximum torsional rotation at point A is

$\phi_A = -[(TL/JG)_{CD} + (TL/JG)_{BC} + (TL/JG)_{AB}]$

$\Rightarrow -\pi/180 = -[(15 \times 12) \times (6 \times 12) + (5 \times 12) \times (5 \times 12) + (10 \times 12) \times (4 \times 12)]/(J\times12000) = -1.86/J$

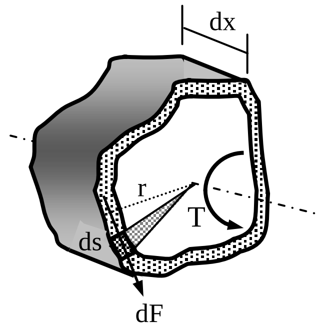
$\Rightarrow J = \pi d^4/32 = 106.57\ in^4 \Rightarrow d = 5.74''$

Torsion of Closed Thin-Walled Sections

Thin-walled sections have wall thickness much smaller than its other dimensions. Closed thin-walled sections are widely used in structures subjected to torsional moments because compared to other sections, they can resist torsional stress and deformations more efficiently.

Torsional Stress and Rotation of Thin-Walled Section

Thin-walled sections can be analyzed for torsion using somewhat similar assumptions as for circular sections; i.e., plane sections remain plane, shear strains are small. However, since the thickness of the section is very small, the shear stresses remain almost constant across the thickness instead of varying linearly from the center of rotation.



But the shear stress does not remain constant throughout the perimeter of the section. Instead the shear flow q, which is the shear force per unit length (given by shear stress ‘τ’ times the wall thickness ‘t’; i.e., q = τ t) is constant.

$\therefore$ Shear force on a small element of length ds is dF = q ds ……..………...(i)

and the corresponding torque dT = r dF = q r ds ……………….(ii)

Since the area of the small triangular area subtended at the center of rotation is = ½ rds, integrating rds over the perimeter of the area will give twice the entire area enclosed by the section [i.e., 2 A].

$\therefore$ Total torque $T = \int dT = \int q\, r\, ds = q\, 2A \Rightarrow \tau = q/t = T/(2At)$ ……..(iii)

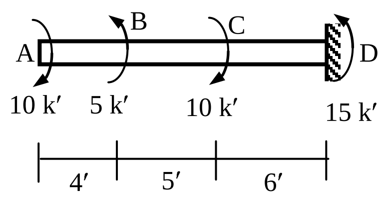
Also, the external energy required by a torque T to cause a twisting rotation $d\phi$ is $= T\, d\phi/2$ ………….(iv)

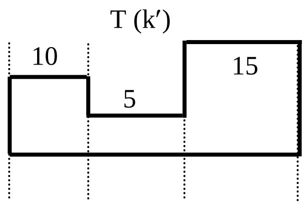
while the corresponding internal energy is $= \int \tau^2/2G\, dV = \int (T^2/8A^2t^2G)\,(dx\; t\; ds)$

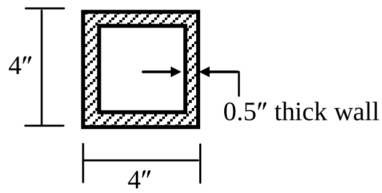
$= (T^2\, dx/8A^2G)(\int ds/t)$ ……………...(v)

Eq. (iv) and (v) $\Rightarrow d\phi/dx = (T/4A^2\, G)(\int ds/t)$; i.e., $J_{eq} = 4A^2/(\int ds/t)$ ……………...(vi)

Example: Considering the torque diagram shown below, calculate the maximum shear stress and torsional rotation for a 4″× 4″ hollow square section with 0.5″ wall thickness [Given: G = 12000 ksi].







The enclosed area A = 3.5″ × 3.5″ = 12.25 in^2

$\therefore \tau_{max} = T/(2At) = (15 \times 12)/(2 \times 12.25 \times 0.5) = 14.69$ ksi

$\therefore$ Also $J_{eq} = 4A^2/\int ds/t = 4(12.25)^2/(4 \times 3.5/0.5) = 21.44$ in^4

$\phi_D - \phi_C = (TL/JG)_{CD} = (15 \times 12) \times (6 \times 12)/(21.44\times12000) = 0.0503$ rad $\Rightarrow \phi_C = -0.0503$ rad $= -2.89°$

$\phi_C - \phi_B = (TL/JG)_{BC} = (5 \times 12) \times (5 \times 12)/(21.44\times12000) = 0.0140$ rad $\Rightarrow \phi_B = -0.0644$ rad $= -3.69°$

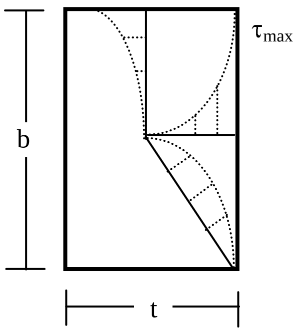
$\phi_B - \phi_A = (TL/JG)_{AB} = (10 \times 12) \times (4 \times 12)/(21.44\times12000) = 0.0224$ rad $\Rightarrow \phi_A = -0.0868$ rad $= -4.97°$

Torsion of Rectangular Sections

Since a majority of civil engineering structures consist of rectangular or assembly of rectangular sections, the study of torsional behavior of such sections is important. However, when subjected to torsional moment, rectangular sections do not behave like circular or thin-walled sections, due to the warping deformations accompanying their response.

Torsional Stress and Rotation of Rectangular Section

The torsional response of these sections cannot be derived using the simple methods of Strength of Materials, and requires concepts of Theory of Elasticity instead, which is beyond the scope of this course. Therefore, only the final expressions for torsional stress and rotation are shown here.



For a (b×t) rectangular section (with $b \geq t$) subjected to torsional moment T, the maximum shear stress $\tau_{max} = T/(\alpha bt^2)$ …..……….…...(i)

and the 'equivalent polar moment of inertia' for torsional rotation is

$J_{eq} = \beta bt^3$ …….……….…...(ii)

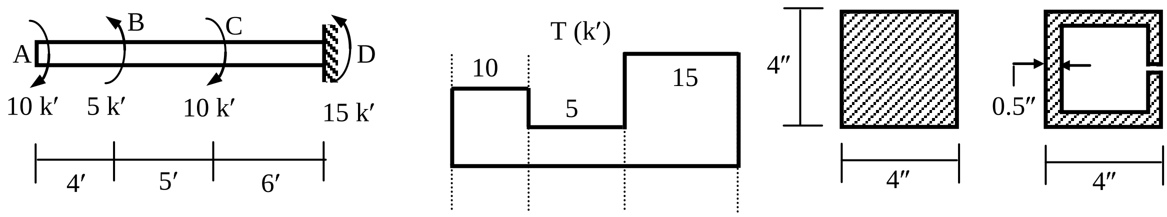
$\therefore$ Torsional rotation for a uniform section of length L is

$\phi = TL/(\beta bt^3)G$ …….……….…..(iii)

The constants α and β in Eqs. (i) and (ii) are non-dimensional parameters and depend on the ratio b/t, as shown in the following table.

b/t	1.0	1.5	2.0	3.0	6.0	10.0	∞
α	0.208	0.231	0.246	0.267	0.299	0.312	0.333
β	0.141	0.196	0.229	0.263	0.299	0.312	0.333

Example: Considering the torque diagram shown below, calculate the maximum shear stress and torsional rotation for a (a) 4″× 4″ solid section (b) 4″× 4″× 0.5″ open section [Given: G = 12000 ksi].



(a) For the 4″× 4″ solid section, $\alpha = 0.208$, $\beta = 0.141$

$\therefore \tau_{max} = T/(\alpha bt^2) = (15 \times 12)/(0.208 \times 4 \times 4^2) = 13.52$ ksi

$\therefore$ Also $J_{eq} = \beta bt^3 = 0.141 \times 4 \times 4^3 = 36.10$ in^4

$\phi_D - \phi_A = \{(15\times12)\times(6\times12) + (5\times12)\times(5\times12) + (10\times12)\times(4\times12)\}/(36.10\times12000) = 0.0515$ rad

$\Rightarrow \phi_A = -0.0515$ rad $= -2.95°$

(b) The 4″× 4″× 0.5″ open section is torsionally equivalent to a rectangular section of size 16″× 0.5″

$\therefore b/t = 32 \Rightarrow$ Both α and β can be assumed to be $\cong 0.333$

$\therefore \tau_{max} = T/(\alpha bt^2) = (15 \times 12)/(0.333 \times 16 \times 0.5^2) = 135$ ksi

$\therefore J_{eq} = \beta\, bt^3 = 0.333 \times 16 \times 0.5^3 = 0.667$ in^4

$\phi_D - \phi_A = \{(15\times12) \times (6\times12) + (5\times12) \times (5\times12) + (10\times12) \times (4\times12)\}/(0.667 \times 12000) = 2.79$ rad

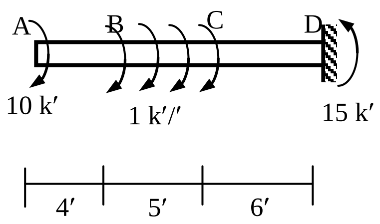
$\Rightarrow \phi_A = -2.79$ rad $= -160°$

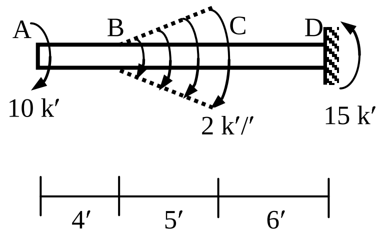
Obviously, common structural materials cannot survive such a large shear stress and angular rotation.

Distributed Torsion and Torsional Rotation

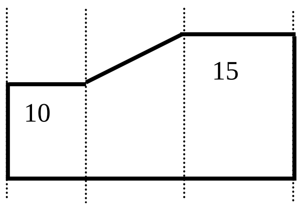
Since the loads on civil engineering (or other) structures are often distributed over a length or over an area, torsional moments on such structures seldom work as concentrated at one section. The torque diagram and calculation of torsional rotation in such cases require special attention.

Distributed Torsion

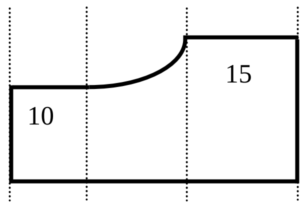




Case 1: T_1 (k′)



Case 2: T_2 (k′)



Example: Considering the torque diagrams shown above, calculate the maximum torsional rotation for a solid circular section of 4″-diameter [Given: G = 12000 ksi].

The polar moment of inertia of the circular section is $J = \pi(4)^4/32 = 25.13$ in^4

Using $\phi_D = 0$, the maximum torsional rotation at point A is

(a) $\phi_A = -\,[(TL/JG)_{CD} + \int_{BC}(T/JG)\,dx + (TL/JG)_{AB}]$

$= -\,[(15 \times 12) \times (6 \times 12) + \{(10 + 15)/2 \times 12\} \times (5 \times 12) + (10 \times 12) \times (4 \times 12)]/(25.13 \times 12000)$

$= -0.0919$ rad $= -5.27°$

(b) $\phi_A = -\,[(TL/JG)_{CD} + \int_{BC}(T/JG)\,dx + (TL/JG)_{AB}]$

$= -\,[(15 \times 12) \times (6 \times 12) + \{(10 + 5/3) \times 12\} \times (5 \times 12) + (10 \times 12) \times (4 \times 12)]/(25.13 \times 12000)$

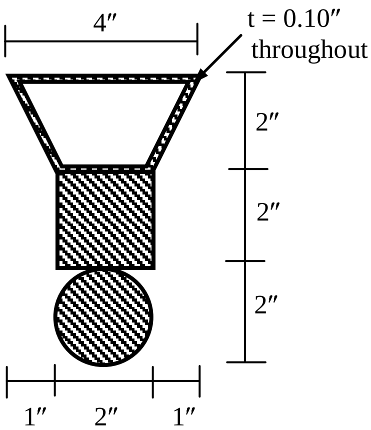
$= -0.0900$ rad $= -5.15°$

Composite and Variable Cross-sections

Composite Sections

Instead of the simple sections (i.e., circular, thin-walled and rectangular), structures subjected to torsion may be sections of arbitrary shape or composite sections made up of two or more simple sections. While arbitrary shaped sections can only be dealt numerically, composites of two or more simple sections can be solved more conveniently. The basic assumption for solving this type of problems is that when subjected to torsion, the section rotates as a rigid body; i.e., $\phi = TL/JG$ is valid for each part of the section so that $\phi = T_1L/J_1G_1 = T_2L/J_2G_2$, etc. Therefore, if G is constant, $T_1/J_1 = T_2/J_2 = \ldots\ldots$..etc; i.e., the torque taken by different parts of the section is proportional to their J values.

Example: Calculate the magnitude and location of the maximum shear stress in the compound section shown below when subjected to a torque of 10 k-ft.



The composite section consists of a thin-walled, a rectangular and a circular section. If the torques taken by them are T_1, T_2 and T_3 respectively, then $T_1/J_1 = T_2/J_2 = T_3/J_3 = T/(J_1 + J_2 + J_3)$

Here, area enclosed by section1, $A = \{(2 + 4)/2\}\times 2 = 6.0$ in^2

$J_1 = 4A^2/\int ds/t = 4\,(6.0)^2/[\{(4 + 2 + 2\sqrt{(2^2 + 1^2)}\}/0.10] = 1.38$ in^4

$J_2 = \beta bt^3 = 0.141\times 2 \times (2)^3 = 2.26$ in^4

$J_3 = \pi d^4/32 = \pi \times 2^4/32 = 1.57$ in^4

$\therefore T_1/1.38 = T_2/2.26 = T_3/1.57 = 10/(1.38 + 2.26 + 1.57) = 10/5.20$

$\Rightarrow T_1 = 2.64$ k-ft, $T_2 = 4.34$ k-ft, $T_3 = 3.02$ k-ft

$\therefore \tau_{max(1)} = T_1/(2At) = 2.64 \times 12/(2 \times 6.0 \times 0.1) = 26.43$ ksi

$\tau_{max(2)} = T_2/(\alpha bt^2) = 4.34 \times 12/(0.208 \times 2 \times 2^2) = 31.28$ ksi

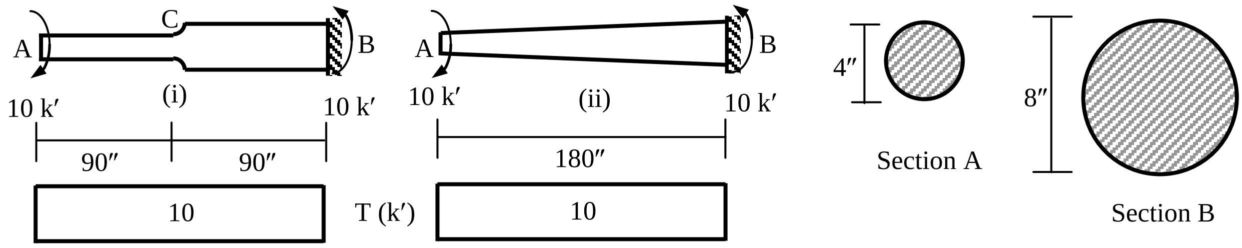
$\tau_{max(3)} = 16\,T_3/(\pi d^3) = 16 \times 3.02 \times 12/(\pi \times 2^3) = 23.07$ ksi

$\therefore \tau_{max} = 31.28$ ksi, at the midpoints of rectangle's sides

Variable Sections

The shape and/or size of the cross-sectional area of a structure subjected to torsional moment may vary over its length. The problem is easier to handle if it varies at particular locations only, although the possibility of stress concentration needs special attention and requires the variations in sections to be smooth or well rounded-off. However, if the variation is gradual over a considerable length, torsional rotation can only be calculated by integration.

Example: Considering the torque diagram shown below, calculate the torsional rotation at A if the cross-section diameter varies from 4″ at A to 8″ at B [Given: G = 12000 ksi].



(a) Neglecting the effect of stress concentration

$\phi_B - \phi_A = (TL/JG)_{AC} + (TL/JG)_{BC} = \{(10\times12)\times90/(\pi\times4^4/32) + (10\times12)\times90/(\pi\times8^4/32)\}/12000$

$\Rightarrow \phi_A = -0.0380$ rad $= -2.18°$

(b) The diameter of the circular section at a distance x from end A is $d(x) = 4 + 4x/180$

$\therefore$ Polar moment of inertia $J(x) = \pi\,(4 + 4x/180)^4/32$

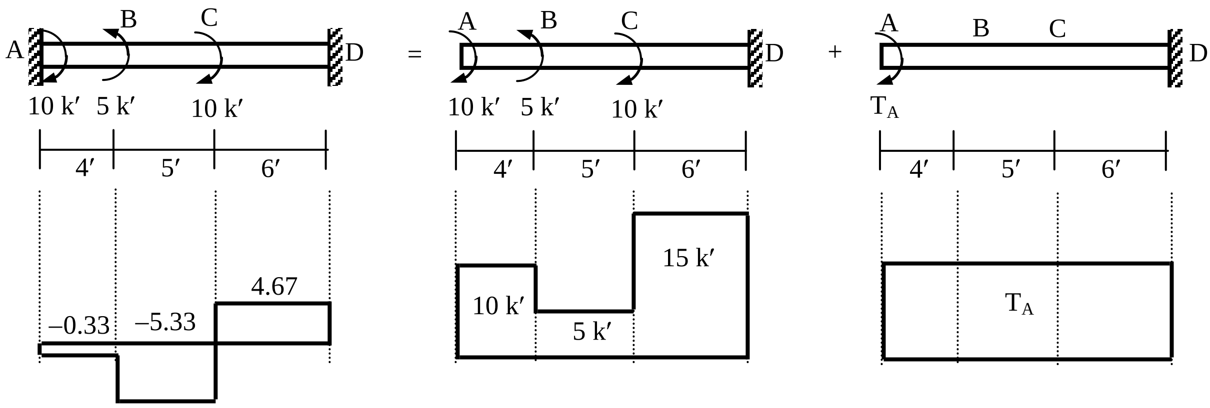
$\therefore \phi_B-\phi_A=\int(T/JG)dx = (10\times12)/(12000)\int dx/(\pi(4+4x/180)^4/32) = (0.01)\,(-180/4)\,\{(1/8)^3-(1/4)^3\}/3/(\pi/32)$

$\Rightarrow \phi_A = -0.0209$ rad $= -1.20°$

Statically Indeterminate Problems on Torsion

A large portion of civil engineering structures is statically indeterminate; i.e., they cannot be analyzed by statics alone. For purely torsional problems, since there is effectively only one equation of statics (i.e., $\sum M_x = 0$), a statically indeterminate structure has another extra unknown. Determining such unknowns require knowledge of displacements; i.e., torsional rotation.

<u>Example</u>: Considering the statically indeterminate torsional problem shown below, calculate
(a) the maximum torsional stress and torsional rotation for a solid circular section of 4″ diameter
(b) the required diameter of a solid circular section if the maximum allowable torsional stress is 10 ksi and allowable torsional rotation is 1° [Given: G = 12000 ksi].



Final Torque Diagram (k′)
Case 1
Case 2

(a) The polar moment of inertia of the circular section is $J = \pi(4)^4/32 = 25.13\ in^4$
The problem is statically indeterminate because the torsional moments at joints A and D are both unknown.
This problem is divided into two statically determinate problems, namely Case 1 and 2.
Using $\phi_D = 0$, the maximum torsional rotation at point A is

Case1: $\phi_{A1} = -[(TL/JG)_{CD} + (TL/JG)_{BC} + (TL/JG)_{AB}]$
$= -[(15 \times 12) \times (6 \times 12) + (5 \times 12)\times (5 \times 12) + (10 \times 12) \times (4 \times 12)]/(25.13 \times 12000)$
$= -0.0740$ rad

Case2: $\phi_{A2} = -T_A L/JG = -(T_A \times 12) \times (15 \times 12)/(25.13 \times 12000) = -0.00716\ T_A$

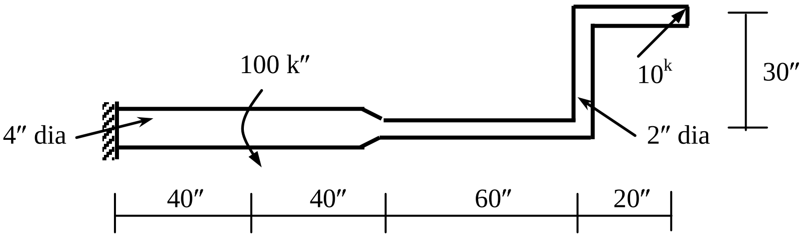
Adding Case1 and 2 $\Rightarrow \phi_{A1} + \phi_{A2} = \phi_A = -0.0740 - 0.00716\ T_A = 0$
$\Rightarrow T_A = -10.33$ k′, from which the final torque diagram can be plotted as shown.

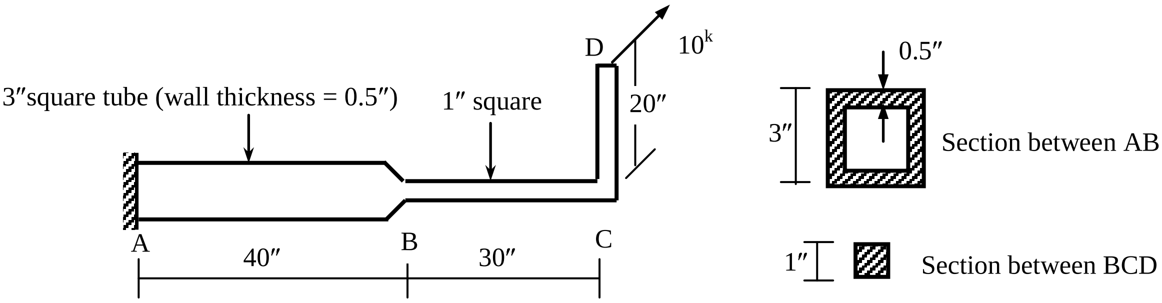
$\therefore$ Using the final torque diagram,
The maximum stress, $\tau_{max} = \tau_{BC} = (Tc/J)_{BC} = (5.33\times12) \times 2/\ 25.13 = 5.09$ ksi, and
The maximum rotation, $\phi_{max} = \phi_C = -(TL/JG)_{CD} = -(4.67\times12) \times (6\times 12)/(25.13\times12000) = -0.0134$ rad
$= -0.766°$

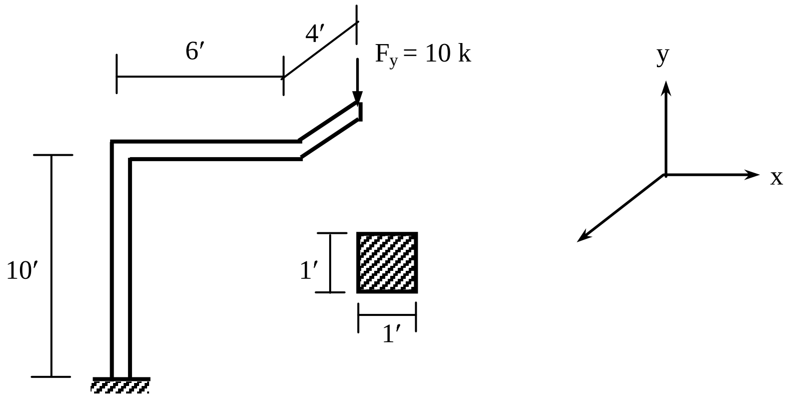
(b) If J is the polar moment of inertia of the circular section, the final torque diagram $\Rightarrow$
The maximum stress, $\tau_{max} = \tau_{BC} = (Tc/J)_{BC} = (16T/\pi d^3)_{BC} = (16\times5.33\times12)/\pi d^3 = (325.95/d^3)$ ksi, and
The maximum rotation, $\phi_{max} = \phi_C = -(TL/JG)_{CD} = -(4.67\times12) \times (6\times 12)/(J\times12000) = -(0.336/J)$ rad
$\therefore \tau_{max} = 10$ ksi $\Rightarrow 325.95/d^3 = 10 \Rightarrow d = 3.19″$
$\therefore \phi_{max} = 1° = \pi/180$ rad $\Rightarrow 0.336/J = \pi/180 \Rightarrow J = \pi d^4/32 = 19.25\ in^4 \Rightarrow d = 3.74″$
$\therefore$ Required diameter, d = 3.74″

Practice Problems on Torsion

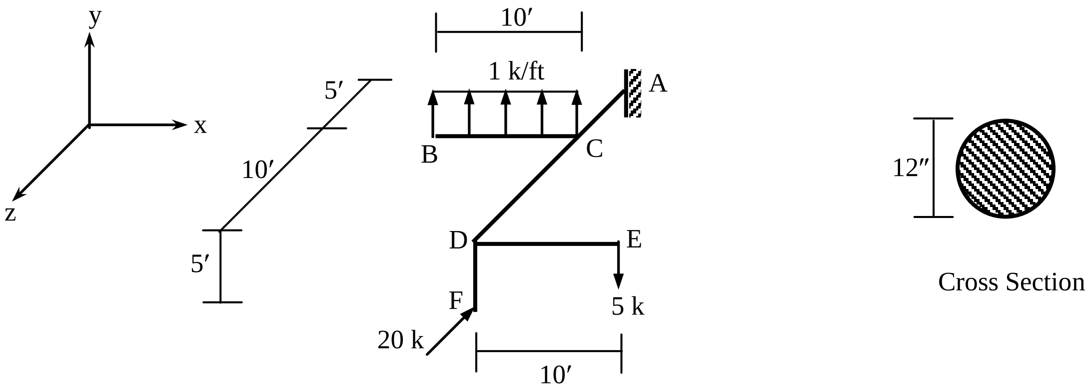
1~3. Calculate the maximum shearing stress and torsional rotation in the structures shown below (Neglect stress concentration) [Given: G = 12000 ksi].



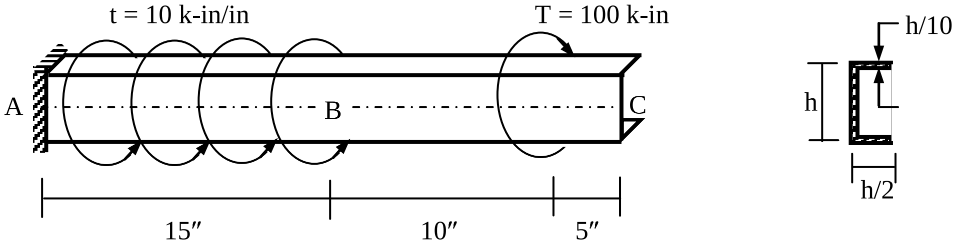




4. For the structure shown below, draw the torque diagram of member ACD and calculate its maximum torsional shear stress and torsional rotation [Given: G = 1000 ksi and the cross-section is a 12″ circle].

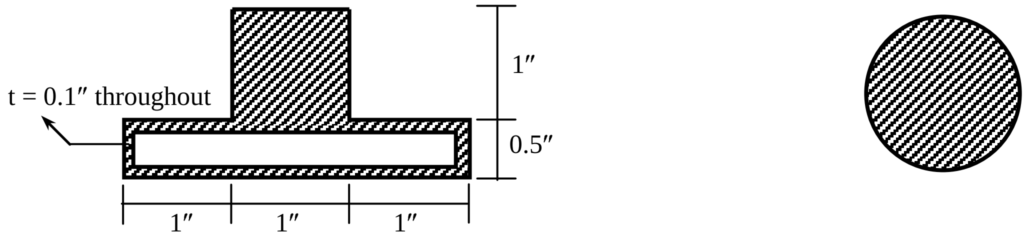


5. Calculate the required depth (h) of the channel section shown below if the allowable shear stress in ABC is 10 ksi and the allowable angle of twist is 1° [Given: G = 12000 ksi].

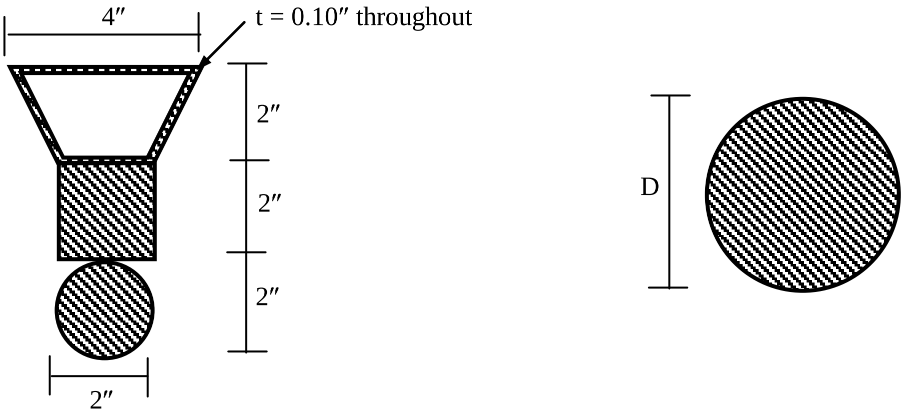


6. Calculate the torsional stiffnesses (i.e., torque required to produce unit rotation per length) of the two thin-walled sections shown below [t<< b].

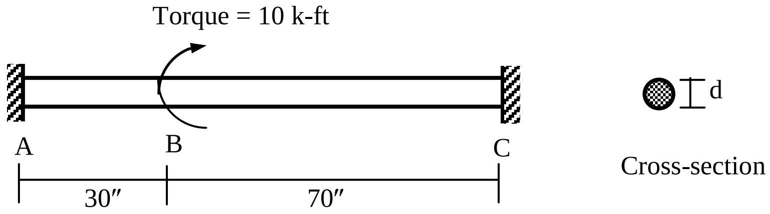
7. The compound section shown below is to be replaced by a circular section so that the torsional stiffness (torque/rotation) remains the same. Calculate the required diameter of the circular section.



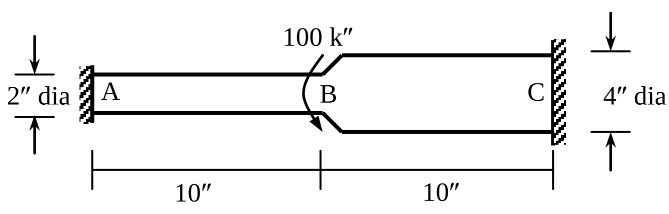
8. Calculate the magnitude and location of the maximum shear stress in the compound section shown below when subjected to a torque of 10 k-ft.
   Also calculate the diameter (D) of the circular section that has the same maximum shear stress when subjected to the same torque.



9. Calculate the required diameter 'd' of the circular rod ABC shown in the figure below if allowable shear stress is 20 ksi. For the diameter 'd', calculate the maximum angle of twist in the rod [G = 12000 ksi].



10. Calculate the torsional shear stress at A and the torsional rotation at B for the circular rod (of non-uniform diameter) shown below (Neglect stress concentration) [Given G = 12000 ksi].



Recent Exam Problems on Torsion

1. For members *ab* and *bcd* (each weighing 0.50 k) of the frame *abcd* loaded as shown in Fig. 1,
   (i) draw the torque diagram,
   (ii) calculate the maximum torsional shear stress and maximum torsional rotation
   [Given: Shear Modulus = 12000 ksi].

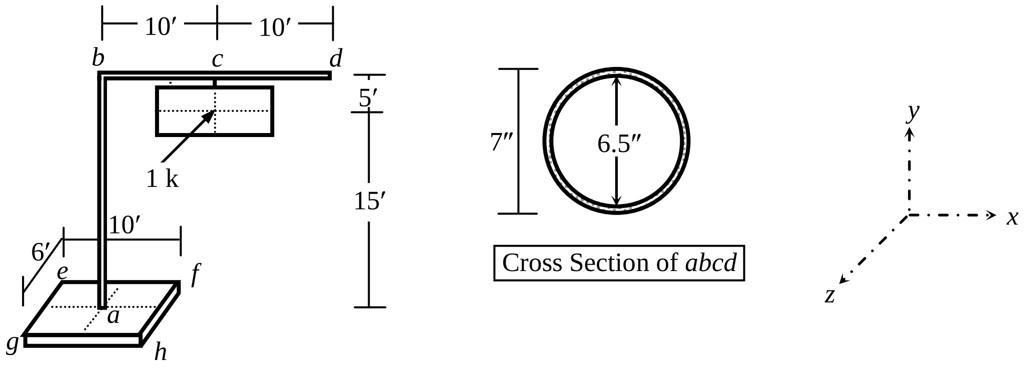
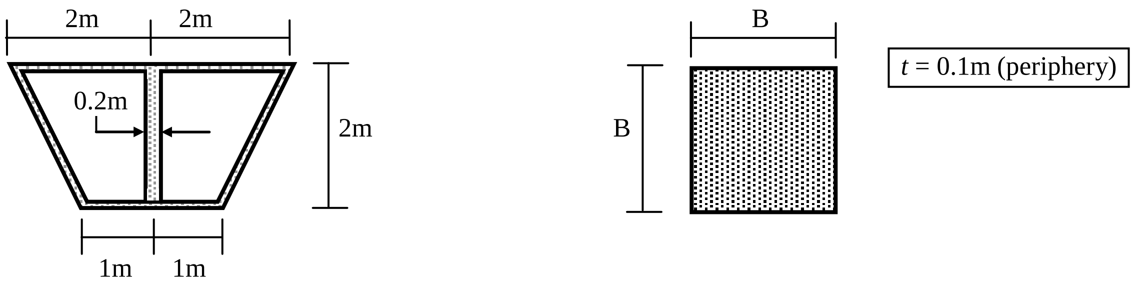


Fig. 1

2. If the compound section shown in Fig. 2 is subjected to a 10 kN-m torque, calculate the
   (i) magnitude of maximum shear stress in the section
   (ii) depth and width (B) of the square section that has the same maximum shear stress when subjected to the same torque.



Compound Section

Square Section

Fig. 2

3. For the rotating shaft ABCD shown in Fig. 3
   (i) calculate the torque T_0 required to make the torsional rotation at B equal to zero,
   (ii) draw the corresponding torque diagram and calculate the maximum shear stresses at A and D

   [Given: Shear Modulus = 12000 ksi].

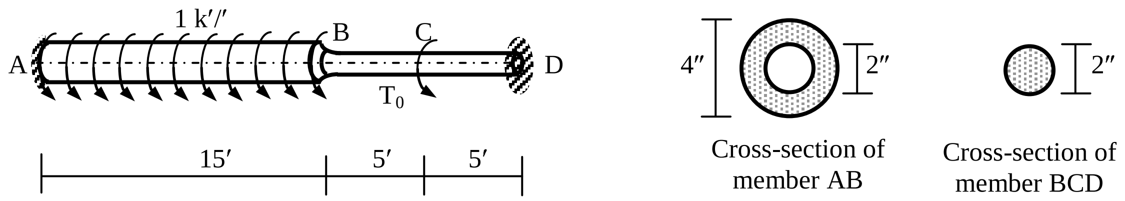


Fig. 3

Stress Combination and Combined Normal Stress

Stress in general is broadly classified as normal stress and shear stress. Normal stresses act perpendicular to the plane (e.g., axial stress and flexural or bending stress), while shear stresses act parallel to the plane (e.g., direct shear, flexural shear and torsional shear stresses). The following table shows the equations for different types of stresses and their validity.

	Type of Stress	Equation	Validity
Normal Stress	Axial	$\sigma = P/A$	All sections
	Flexural	$\sigma = My/I$	All sections
Shear Stress	Direct Shear	$\tau = V/A$	All sections
	Flexural Shear	$\tau = VQ/It$	All sections
	Torsional Shear	$\tau = Tc/J$	Circular sections
		$\tau = T/(2A\ t)$	Thin-walled sections
		$\tau = T/(\alpha b t^2)$	Rectangular sections

Practical problems on stress analysis, including those in civil engineering, almost always consist of a variety of stress conditions. For example, in a typical building structure, beams and slabs are subjected to significant flexural and shear stresses due to vertical loads, while columns and footings have significant axial and flexural stresses (and possibly shear stresses mainly due to lateral loads).

This discussion focuses on combination of normal stress with normal stress and shear stress with shear stress. The more general topic of stress combination (normal stress with shear stress) is covered in the transformation of stresses.

Combination of Axial and Bending Stress

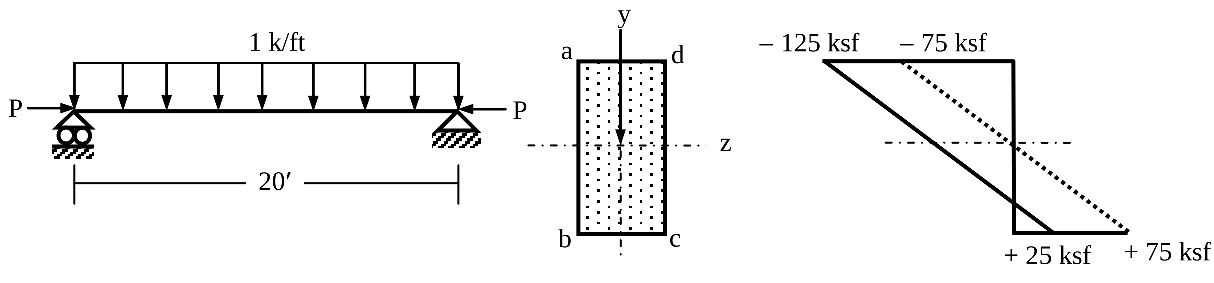
Combination of normal stresses in the form of axial and bending stress is quite common in practical situations; e.g., in beams, columns and footings in civil engineering structures. In this case, the combined normal stress is simply the algebraic sum of two stresses, given by

$$\sigma_z\,(x,y) = P/A - M_z\,y/I_z \qquad \text{…………………(i)}$$

where P is the tensile force and M_z is the bending moment on a cross-section whose area is A, and moment of inertia about z-axis is I_z. A very common example of combination of axial and bending stress is the case of pre-stressed concrete, where a flexural member (one under bending stress) is subjected to sufficient compressive stress in order to reduce or avoid tension in concrete, which is relatively weak in tension.

Example: Calculate the maximum stresses at the midspan section of a 20′ long simply supported rectangular beam of (1′×2′) section under a uniformly distributed load of 1 k/ft, if it is subjected to an additional pre-stressing compressive force of P = 100 kips.

Also calculate the pre-stressing force P_0 necessary to avoid tension in the section altogether.



With b = 1′, h = 2′

$\Rightarrow$ Cross-sectional Area A = 1×2 = 2 ft^2, Moment of Inertia about z-axis $I_z = 1\times2^3/12 = 0.667$ ft^4

Maximum midspan bending moment $M_z = 1 \times 20^2/8 = 50$ k-ft

$\therefore \sigma_{a,d} = -\,100/2.0 - 50 \times 1/(0.667) = -50 - 75 = -125$ ksf, $\sigma_{b/c} = -50 + 75 = +\,25$ ksf

If the necessary pre-stressing force to avoid tension in the section is P_0, then

$\sigma_{t(max)} = \sigma_{b.c} = -P_0/2.0 + 75 = 0 \Rightarrow P_0 = 150$ kips

Combination of Bending Stresses: Biaxial Bending

Biaxial (involving two axes) bending is the bending of a cross-section about two axes of rotation and often deals with bending about the centroidal axes. For an area subjected to biaxial moments M_z and M_y about the centroidal z and y axes, the compressive stress at a point with coordinates (z, y) is given by

$$\sigma_x(z, y) = M_z\, y/I_z + M_y\, z/I_y \qquad \text{..................(i)}$$

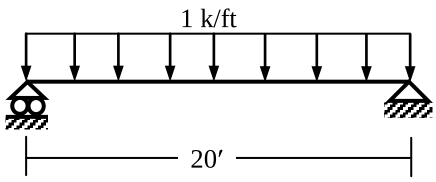
where I_z and I_y are the moments of inertia of the cross-section.

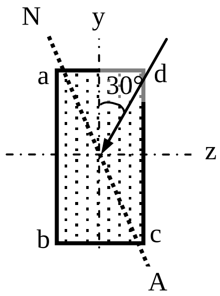
Therefore, the neutral axis is defined by the equation $\sigma_x(z,y) = 0 \Rightarrow M_z\, y/I_z + M_y\, z/I_y = 0$

$$\Rightarrow y = -(M_y/M_z)\,(I_z/I_y)\, z \qquad \text{.................(ii)}$$

Also, σ_{max} at the four corners of a rectangular section is given by $\sigma_{max} = \pm\, 6M_z/bh^2 \pm 6M_y/hb^2$ …..……....(iii)

<u>Example</u>: Calculate the maximum bending stresses at the four corners of the midspan section of a 20′ long simply supported rectangular beam of (1′×2′) section under a uniformly distributed load of 1 k/ft, inclined at 30° with vertical.





With b = 1′, h = 2′
$I_z = 1\times2^3/12 = 0.667$ ft^4
$I_y = 2\times1^3/12 = 0.167$ ft^4
Maximum midspan bending moments
$M_z = (1 \cos 30°)\, 20^2/8 = 43.3$ k-ft
$M_y = (1 \sin 30°)\, 20^2/8 = 25$ k-ft

$\therefore \sigma_a = -43.3 \times 1/(0.667) + 25 \times 0.5/(0.167) = -64.95 + 75 = +\, 10.05$ ksf, $\sigma_b = +\, 64.95 + 75 = +\, 139.95$ ksf,
$\sigma_c = +\, 64.95 - 75 = -10.05$ ksf, $\sigma_d = -64.95 - 75 = -139.95$ ksf

Equation of the neutral axis is,

$$y = -(25/43.3) \times (0.667/0.167)\, z = -2.31\, z$$

which is a straight line through origin as indicated by the line NA.

Combination of Axial and Biaxial Bending Stress

Among civil engineering structures or structural elements, columns and footings often provide common examples of situations involving combination of axial stress and uniaxial or biaxial bending stresses.

Combination of Axial and Bending Stresses in Footings

A loading situation with axial and biaxial bending stress can be due to concentric axial force accompanied by moments about the centroidal axes [Fig. 1(a)] or a biaxially eccentric axial force [Fig. 1(b)].

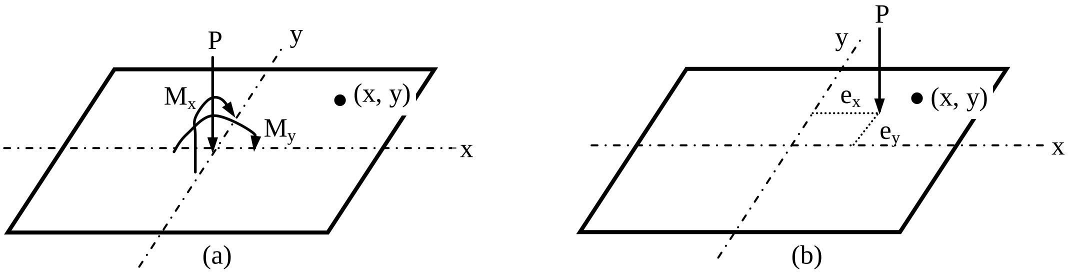


Fig. 1: Footing subjected to (a) Concentric axial load and biaxial bending, (b) Biaxially eccentric axial load

For a cross-sectional area subjected to a concentric compressive force P and biaxial moments M_x and M_y about the centroidal x and y axes [Fig. 1(a)], the compressive stress at a point with coordinates (x, y) is

$$\sigma_z (x,y) = - P/A - M_x\, y/I_x - M_y\, x/I_y \qquad \text{………………….(i)}$$

where A, I_x and I_y are the area and moments of inertia of the cross-section. For a biaxially eccentric compressive load P located at a point (e_x, e_y) in the coordinate axes; i.e., with eccentricities e_y and e_x about the x and y axes respectively as shown in Fig. 1(b), the biaxial bending moments are $M_x = Pe_y$ and $M_y = Pe_x$ and the compressive stress at (x, y) is given by

$$\sigma_z (x,y) = - P/A - Pe_y\, y/I_x - Pe_x\, x/I_y \qquad \text{………………….(ii)}$$

Kern of a Footing Area

Since the soil below foundation can hardly take any tension, it is important that the force P be applied on the footing in a manner to ensure the stresses below the entire area under the footing remain compressive. The zone within which the load is to be applied is called the *kern* of the area. A line of action of the applied load that ensures compression is given by $\sigma (x,y) \leq 0$

$$\therefore - P/A - Pe_y\, y/I_x - Pe_x\, x/I_y \leq 0 \;\Rightarrow\; 1/A + e_x\, x/I_y + e_y\, y/I_x \geq 0 \qquad \text{………………….(iii)}$$

For a rectangular area (b×h), A = bh, $I_x = bh^3/12$, $I_y = hb^3/12$, Eq. (iii) ⇒

$$1 + 12\, e_x\, x/b^2 + 12\, e_y\, y/h^2 \geq 0 \qquad \text{………………….(iv)}$$

∴ To ensure compressive stress at the corner point where (x, y) = (b/2, h/2),

$$1 + 6\, e_x/b + 6\, e_y/h \geq 0 \Rightarrow e_x/(b/6) + e_y/(h/6) \geq -1 \qquad \text{………………….(v)}$$

∴ Similarly, to ensure compressive stress at corner points (b/2,–h/2), (–b/2, h/2) (–b/2,–h/2) ⇒

$$-1 \leq e_x/(b/6) + e_y/(-h/6) \leq 1;\; e_x/(b/6) + e_y/(h/6) \leq 1 \qquad \text{……..……(vi)~(viii)}$$

Plotting the lines defined by Eqs. (v)~(viii) defines a zone (called *kern*, Fig. 2), a parallelogram with diagonals b/3 and h/3, within which the load must act in order to ensure compressive stresses at the four corners of the footing, thereby ensuring an entire area under the footing free of tensile stresses.

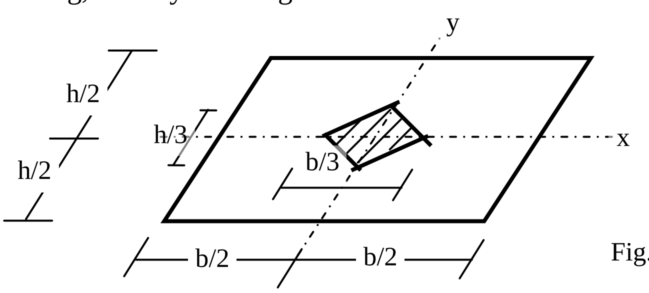


Fig. 2: Kern of a rectangular area

Combination of Shear Stresses for Helical Spring

Helical springs provide an important example of combining shear stresses. These springs are used in several engineering structures and equipment as load transferring elements or shock absorbers.

Combined Shear Stresses in Helical Springs

Helical springs made of rods or wires of circular cross-section (as shown in Fig. 1) may be analyzed in the elastic range by the superposition of shearing stresses. One important assumption needed here is that any one coil of such a spring will be assumed to be in a plane, which is nearly perpendicular to the axis of the spring. This assumption can be made if the adjoining coils are close enough.

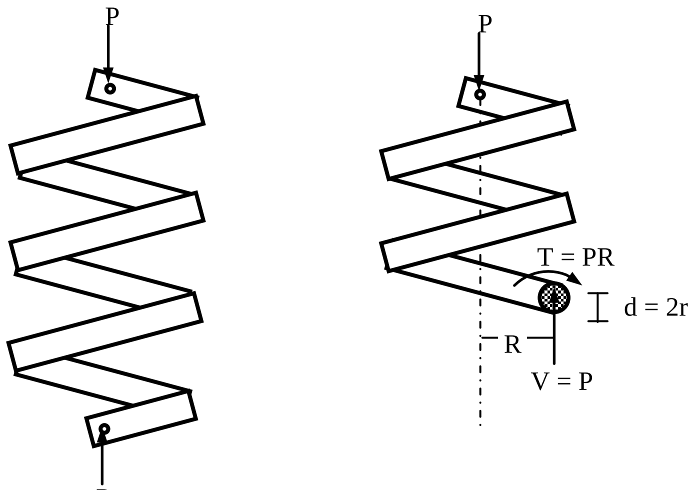


Fig. 1: Forces acting on a closely coiled Helical Spring

Therefore the forces acting on any section of the spring are (i) Shearing force V = P, (ii) Torque T = PR; where P = force applied along the axis of the spring, R = distance of the spring axis to the centroid of the coil's cross-section. The maximum shearing stress at an arbitrary section of the spring is found by superposing the direct and torsional shear stresses, using J = Polar moment of inertia of coil section = $\pi d^4/32 = \pi r^4/2$.

$\therefore$ By superposing, one obtains $\tau_{max} = \tau_{direct} + \tau_{torsion} = P/A + Tr/J = P/A\,(1 + 2R/r)$ ……..…....(i)

Deflection and Stiffness of Helical Springs

In Fig. 1 (c), $\sin\alpha = BC/BD = EG/DG$ …………(ii)

If $EG = d\Delta$ is the differential deflection of the small section AB (of length dx), and using $DG = BD\,(d\phi)$, Eq. (ii) $\Rightarrow R/BD = (d\Delta)/BD\,(d\phi) \Rightarrow d\Delta = R\,(d\phi)$ ……..…(iii)

Also, $d\phi = T(dx)/JG = PR\,(dx)/JG$; $\therefore$ The deflection of a helical spring can be obtained (neglecting the deflection due to direct shear stress, which is normally small) by using the following relationship

$$d\Delta = PR^2\,dx/JG \Rightarrow \Delta = PR^2L/JG \qquad \text{……..…(iv)}$$

where L = length of the spring's rod and G = shearing modulus of elasticity (also called the modulus of rigidity). For a closely coiled spring the length L of the wire may be obtained with sufficient accuracy as $2\pi RN$, where N is the number of live or active coils of the spring.

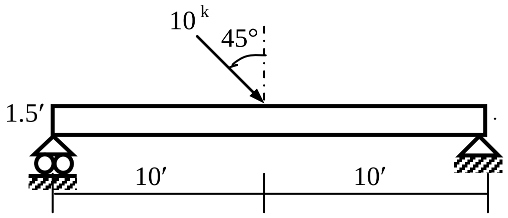
$$\therefore \Delta = 64\,PR^3N/Gd^4 \qquad \text{………..…(v)}$$

This equation can be used to obtain the deflection of a closely coiled helical spring along its axis when subjected to either tensile or compressive force P. The stiffness of a spring, often referred to as spring constant (commonly denoted by k), is defined as the force required to produce unit deflection.

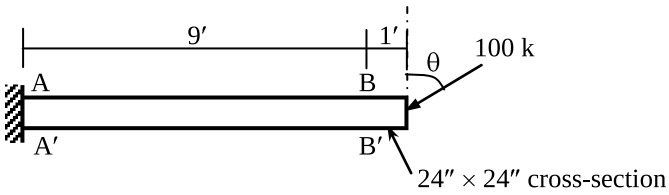
$$\therefore k = P/\Delta = Gd^4/(64R^3N) \qquad \text{…………(vi)}$$

Practice Problems on Combination of Stress

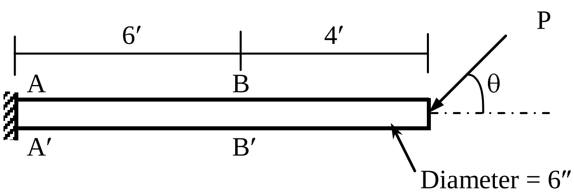
1. Calculate the maximum normal stress in the beam shown below and show the point/points where it occurs [The beam area is rectangular and 1′ wide].



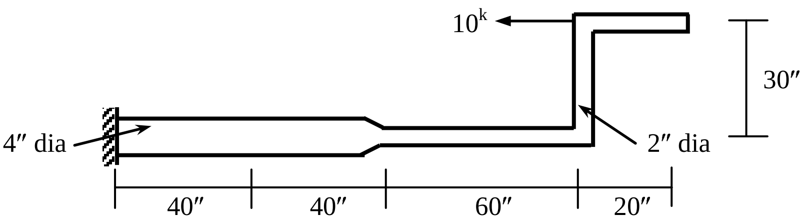
2. In the beam shown below, draw the normal stress-diagram over the sections A-A′ and B-B′ if θ = 60°.



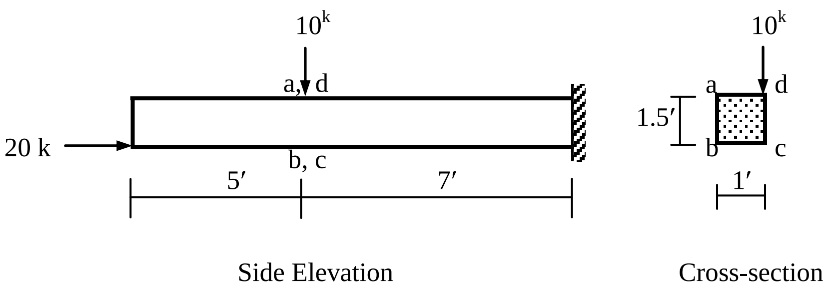
3. In the beam shown below, calculate the maximum allowable value of θ in order to avoid tension in (i) Section A-A′, (ii) Section B-B′.



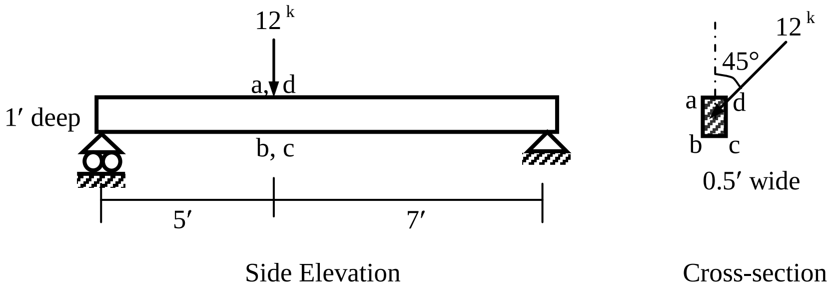
4. Calculate the maximum normal stress in the structure shown below (Neglect stress concentration).



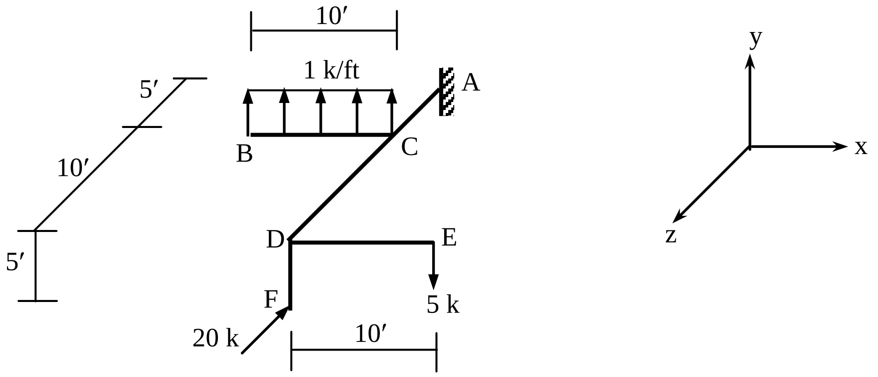
5. Calculate the maximum compound normal stress in the beam shown in the figures below and show the point(s) where it occurs [The beam area is a 1′× 1.5′ rectangle].



6. Calculate the maximum compound normal stress in the beam shown below (subjected to inclined loading) and show the point/points where it occurs [The beam area is a 0.5′× 1′ rectangle].

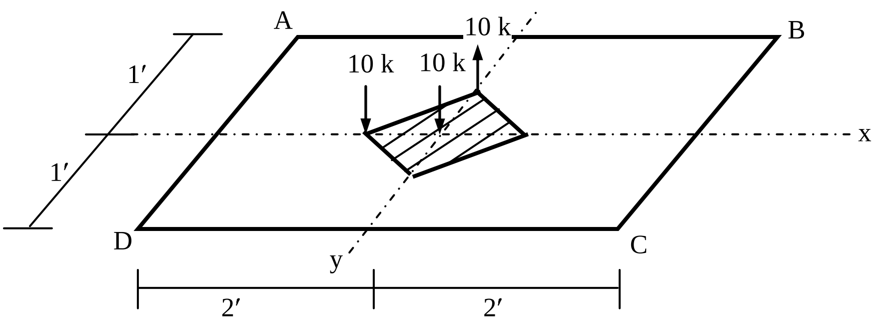


7. For the structure shown below, draw the axial force and bending moment diagram of member ACD and calculate the maximum normal stress in the member.



8. Determine the kern of a solid circular cross section of radius R and show the kern on the section.

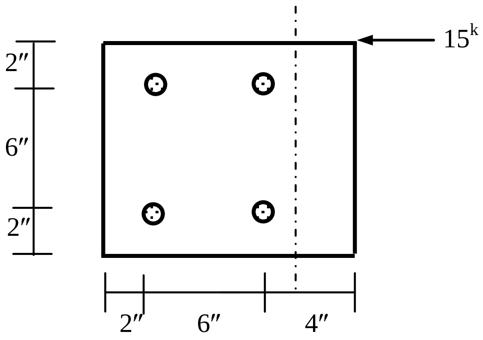
9. The shaded area shown below represents the kern of the rectangular footing ABCD. For the given loads calculate the normal stresses at A, B, C, D and locate the neutral axis.



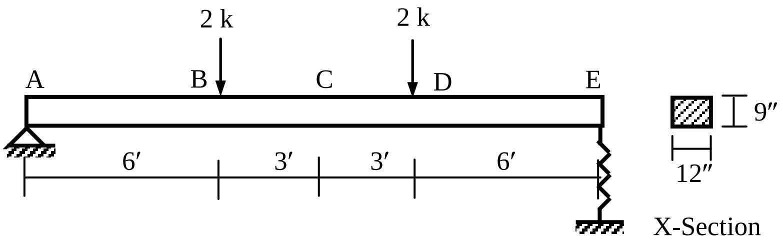
10. Calculate the maximum compound shear stress in the beam described in Problem 5 and show the point(s) where it occurs.

11. In the structure described in Problem 7, calculate the maximum compound shear stress at A.

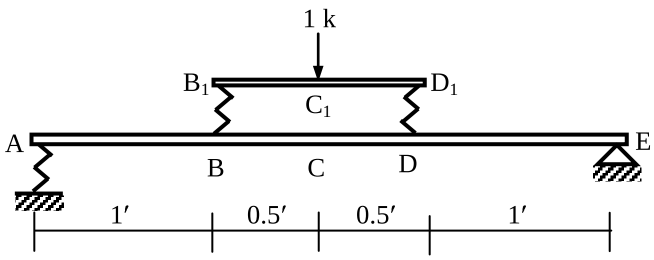
12. Design the connection bolts for shear, under the loading condition shown in the figure below, if the allowable shear stress is 12 ksi.



13. Calculate the deflection at point C for the timber beam ABCDE loaded as shown below, if the spring at E has shear modulus = 12000 ksi, coil diameter = 2″, inside diameter of spring = 8″, number of coils = 8 [Modulus of elasticity of timber = 1500 ksi].



14. In the figure shown below, both $B_1C_1D_1$ and ABCDE are rigid beams. The helical springs at A, B_1 and D_1 have coil diameter = 1 in, average spring radius = 3 in, number of coils = 5 and shear modulus = 12000 ksi. Calculate the deflections at B, D, B_1, D_1 and the combined shear stress for the spring at A.



Transformation of Stresses

The earlier discussions on stress combination dealt only with the superposition of normal stress with normal stress (e.g., axial stress with flexural stress) and shear stress with shear stress (e.g., direct shear stress with torsional shear stress). However, many practical situations require the combination of normal stress with shear stress (e.g., axial and flexural stress with flexural and torsional shear stress).

Besides, the failure of several structural materials (e.g., concrete, steel, timber) due to various types of loading (e.g., tension, compression, shear, bending, torsion) occurs along different surfaces based on material properties and orientation of maximum stress. The choice of these surfaces requires the knowledge about stress transformation, to know the normal and shear stresses along any surface under any particular stress condition.

Equations of Transformation of Stresses

Equations for the transformation of normal and shear stresses on a differential element over any surface to a plane in another orientation is derived here. Fig. 1(a) shows the normal stresses σ_{xx}, σ_{yy} along the x- and y-directions on the vertical and horizontal surfaces respectively and the shear stress τ_{xy} parallel to both the surfaces.

Fig. 1(b) shows the free-body of a small element demonstrating stresses σ_{xx}', τ_{xy}' over a plane of area dA oriented at an angle θ to the original surface (acted upon by stresses σ_{xx} and τ_{xy}).

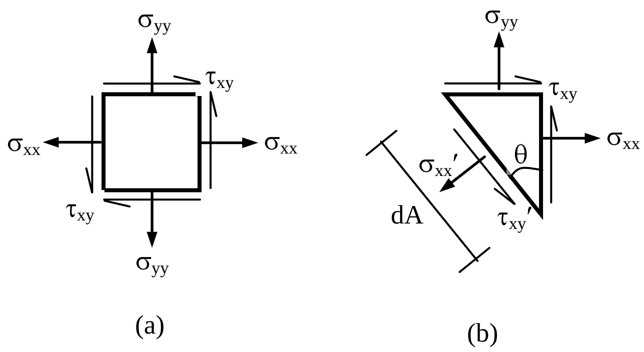


Fig. 1: Transformation of Normal and Shear Stresses

Considering the equilibrium of the element shown in the free-body diagram of Fig. 1(b),

$\sum F_x = 0 \Rightarrow \sigma_{xx}\, dA \cos\theta + \tau_{xy}\, dA \sin\theta - (\sigma_{xx}'\, dA)\cos\theta + (\tau_{xy}'\, dA)\sin\theta = 0$

$\Rightarrow \sigma_{xx}' \cos\theta - \tau_{xy}' \sin\theta = \sigma_{xx}\cos\theta + \tau_{xy}\sin\theta$ ……………………………………..(i)

$\sum F_y = 0 \Rightarrow \tau_{xy}\, dA \cos\theta + \sigma_{yy}\, dA \sin\theta - (\sigma_{xx}'\, dA)\sin\theta - (\tau_{xy}'\, dA)\cos\theta = 0$

$\Rightarrow \sigma_{xx}' \sin\theta + \tau_{xy}' \cos\theta = \tau_{xy}\cos\theta + \sigma_{yy}\sin\theta$ ……………………………………..(ii)

$[(i)\times\cos\theta + (ii)\times\sin\theta] \Rightarrow \sigma_{xx}'(\cos^2\theta + \sin^2\theta) = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\,\tau_{xy}\cos\theta\sin\theta$

$\Rightarrow \sigma_{xx}' = \sigma_{xx}(1+\cos 2\theta)/2 + \sigma_{yy}(1-\cos 2\theta)/2 + \tau_{xy}\sin 2\theta$

$\therefore \sigma_{xx}' = (\sigma_{xx}+\sigma_{yy})/2 + \{(\sigma_{xx}-\sigma_{yy})/2\}\cos 2\theta + (\tau_{xy})\sin 2\theta$ ……………….(iii)

$[(i)\times\sin\theta - (ii)\times\cos\theta] \Rightarrow -\tau_{xy}'(\sin^2\theta + \cos^2\theta) = \sigma_{xx}\cos\theta\sin\theta - \sigma_{yy}\cos\theta\sin\theta + \tau_{xy}(\sin^2\theta - \cos^2\theta)$

$\Rightarrow -\tau_{xy}' = \sigma_{xx}(\sin 2\theta)/2 - \sigma_{yy}(\sin 2\theta)/2 - \tau_{xy}\cos 2\theta$

$\therefore \tau_{xy}' = -\{(\sigma_{xx}-\sigma_{yy})/2\}\sin 2\theta + (\tau_{xy})\cos 2\theta$ ……………….(iv)

<u>Example</u>: If the normal and shear stresses on the element shown in Fig. 1(a) are $\sigma_{xx} = 20$ ksi, $\sigma_{yy} = -10$ ksi and $\tau_{xy} = 15$ ksi, calculate the normal stress σ_{xx}' and shear stress τ_{xy}' on a plane defined by $\theta = -30°$.

Eq. (iii) $\Rightarrow \sigma_{xx}' = (\sigma_{xx}+\sigma_{yy})/2 + \{(\sigma_{xx}-\sigma_{yy})/2\}\cos 2\theta + (\tau_{xy})\sin 2\theta$

$= (20-10)/2 + \{(20+10)/2\}\cos(-60°) + (15)\sin(-60°)$

$= 5 + 15\cos(-60°) + 15\sin(-60°) = -0.49$ ksi

Eq. (iv) $\Rightarrow \tau_{xy}' = -\{(\sigma_{xx}-\sigma_{yy})/2\}\sin 2\theta + (\tau_{xy})\cos 2\theta$

$= -\{(20+10)/2\}\sin(-60°) + (15)\cos(-60°) = 20.49$ ksi

Principal Stresses and Principal Planes

The normal and shear stresses on a plane at an angle θ with a reference plane acted on by normal stresses (σ_{xx}, σ_{yy}) and shear stress τ_{xy} are given by the following expressions

$$\sigma_{xx}' = (\sigma_{xx} + \sigma_{yy})/2 + \{(\sigma_{xx} - \sigma_{yy})/2\} \cos 2\theta + (\tau_{xy}) \sin 2\theta \qquad \ldots\ldots\ldots\ldots\text{(iii)}$$

$$\tau_{xy}' = -\{(\sigma_{xx} - \sigma_{yy})/2\} \sin 2\theta + (\tau_{xy}) \cos 2\theta \qquad \ldots\ldots\ldots\ldots\text{(iv)}$$

These equations can also be written as

$$\sigma_{xx}' = (\sigma_{xx} + \sigma_{yy})/2 + \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]} \cos (2\theta - \alpha) \qquad \ldots\ldots\ldots\ldots\text{(v)}$$

$$\tau_{xy}' = -\sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]} \sin (2\theta - \alpha) \qquad \ldots\ldots\ldots\ldots\text{(vi)}$$

where $\tan \alpha = 2\, \tau_{xy}/(\sigma_{xx} - \sigma_{yy})$

$\therefore$ The maximum and minimum values of normal stress are

$$\left.\begin{aligned} \sigma_{xx(max)} &= (\sigma_{xx} + \sigma_{yy})/2 + \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]};\ \text{when } \theta = \alpha/2,\ \alpha/2 + 180° \\ \sigma_{xx(min)} &= (\sigma_{xx} + \sigma_{yy})/2 - \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]};\ \text{when } \theta = \alpha/2 \pm 90° \end{aligned}\right\} \qquad \ldots\ldots\ldots\text{(vii)}$$

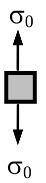
The stresses $\sigma_{xx(max)}$ and $\sigma_{xx(min)}$, also denoted by σ_1 and σ_2, are called the *principal stresses*, while the mutually perpendicular planes they act on, represented by $\theta = \alpha/2$, $\alpha/2 + 180°$ and $\theta = \alpha/2 \pm 90°$, are called the *principal planes*. They represent the maximum tensile and compressive stresses at the corresponding point. These stresses and planes are extremely important in analyzing the failure criteria of structural materials, particularly for brittle materials. An important aspect of the principal planes is that the shear forces on them are zero.

However, for 'yielding' materials or the ones that fail in shear, the maximum values of shear stress and the corresponding planes are also very important

$$\left.\begin{aligned} \tau_{xy(max)} &= \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]};\ \text{when } \theta = \alpha/2 - 45°,\ \alpha/2 + 135° \\ \tau_{xy(min)} &= -\sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]};\ \text{when } \theta = \alpha/2 + 45°,\ \alpha/2 - 135° \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\text{(viii)}$$

Typical Example Problems

The equations (vii) and (viii) can be used to predict the maximum stress and failure surface in several typical stress conditions.



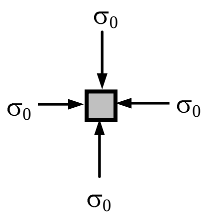
Uniaxial Tension

$\sigma_{xx} = 0$, $\sigma_{yy} = \sigma_0$, $\tau_{xy} = 0 \Rightarrow \tan \alpha = 2 \times 0/(0 - \sigma_0) \Rightarrow \alpha = 180°$

$\therefore \sigma_1 = (0 + \sigma_0)/2 + \sqrt{[\{(0 - \sigma_0)/2\}^2 + (0)^2]} = \sigma_0/2 + \sigma_0/2 = \sigma_0$, when $\theta = 90°, 270°$

and $\sigma_2 = \sigma_0/2 - \sigma_0/2 = 0$, when $\theta = 0°, 180°$

$\therefore \tau_{xy(max)} = \sigma_0/2$, when $\theta = 45°, 225°$; $\tau_{xy(min)} = -\sigma_0/2$, when $\theta = +135°, -45°$



Hydrostatic Compression

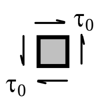
$\sigma_{xx} = -\sigma_0$, $\sigma_{yy} = -\sigma_0$, $\tau_{xy} = 0 \Rightarrow \tan \alpha = 2 \times 0/(-\sigma_0 + \sigma_0) \Rightarrow \alpha$ is indeterminate

$\therefore \sigma_1 = (-\sigma_0 - \sigma_0)/2 + \sqrt{[\{(-\sigma_0 + \sigma_0)/2\}^2 + (0)^2]} = -\sigma_0 + 0 = -\sigma_0$

and $\sigma_2 = -\sigma_0 - 0 = -\sigma_0$

$\therefore \tau_{xy(max)} = 0$; $\tau_{xy(min)} = 0$

Therefore, the normal stress is $-\sigma_0$ and shear stress is zero *on all surfaces*



Pure Shear

$\sigma_{xx} = 0$, $\sigma_{yy} = 0$, $\tau_{xy} = \tau_0 \Rightarrow \tan \alpha = 2\, \tau_0/(0 - 0) \Rightarrow \alpha = 90°$

$\therefore \sigma_1 = (0 + 0)/2 + \sqrt{[\{(0 - 0)/2\}^2 + (\tau_0)^2]} = 0 + \tau_0 = \tau_0$, when $\theta = 45°, 225°$

and $\sigma_2 = 0 - \tau_0 = -\tau_0$, when $\theta = -45°, 135°$

$\therefore \tau_{xy(max)} = \tau_0$, when $\theta = 0°, 180°$; $\tau_{xy(min)} = -\tau_0$, when $\theta = \pm 90°$

Mohr's Circle

The equations for the normal and shear stresses on a plane at angle θ with a reference plane acted on by normal stresses (σ_{xx}, σ_{yy}) and shear stress τ_{xy} have been derived to be

$$\sigma_{xx}' = (\sigma_{xx} + \sigma_{yy})/2 + \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]} \cos (2\theta-\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

$$\tau_{xy}' = -\sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]} \sin (2\theta-\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

where $\tan \alpha = 2\, \tau_{xy}/(\sigma_{xx} - \sigma_{yy})$

These equations can be re-adjusted to the following form

$$\{\sigma_{xx}' - (\sigma_{xx} + \sigma_{yy})/2\}^2 + (\tau_{xy}'-0)^2 = \{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}$$

Eq. (ix), when plotted with σ_{xx}' in x-axis and τ_{xy}' in y-axis, takes the form $(X-a)^2 + (Y-0)^2 = R^2$, which is the equation of a circle with center$(a, 0) = [(\sigma_{xx} + \sigma_{yy})/2, 0]$ and radius $R = \sqrt{[\{(\sigma_{xx}-\sigma_{yy})/2\}^2 + (\tau_{xy})^2]}$. This circle is called *Mohr's Circle*, after Otto Mohr of Germany, who first suggested it in 1895.

Fig. 1 shows a Mohr's Circle with some of its more important features. Among them, the coordinates of the center of the circle = $(a, 0) = [(\sigma_{xx} + \sigma_{yy})/2, 0]$ and radius $R = \sqrt{[\{(\sigma_{xx} - \sigma_{yy})/2\}^2 + (\tau_{xy})^2]}$ have already been mentioned before.

However, the figure also shows that the principal stresses are $\sigma_1 = a + R$, $\sigma_2 = a - R$, while the maximum and minimum shear stresses are $\tau_{max} = R$ and $\tau_{min} = -R$.

Since the center of the circle is at the midpoint of all radial lines, $(\sigma_1 + \sigma_2)/2 = (\sigma_{xx} + \sigma_{yy})/2 = a$

Also from figure, $\tan \phi = (\tau_{xy}-0)/(\sigma_{xx}-(\sigma_{xx} + \sigma_{yy})/2) = 2\tau_{xy}/(\sigma_{xx}-\sigma_{yy}) = \tan \alpha$; $\therefore \phi = \alpha$.

It can also be proved that if the slope of AB with vertical is θ, the coordinates of B = $(\sigma_{xx}', -\tau_{xy}')$.

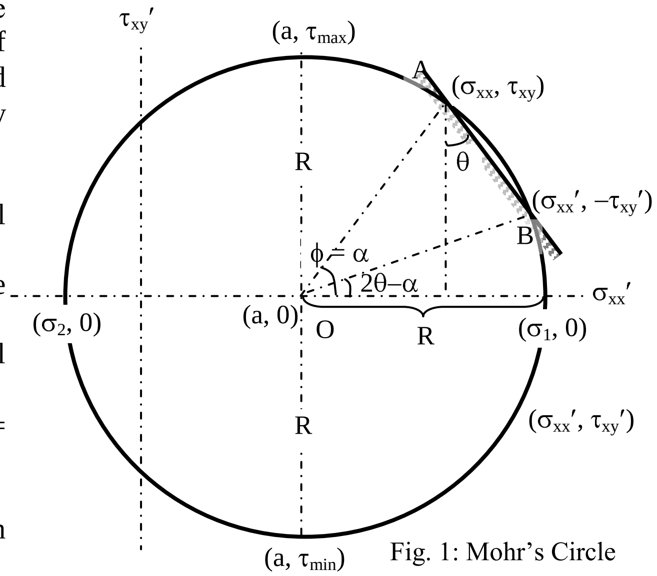


Fig. 1: Mohr's Circle

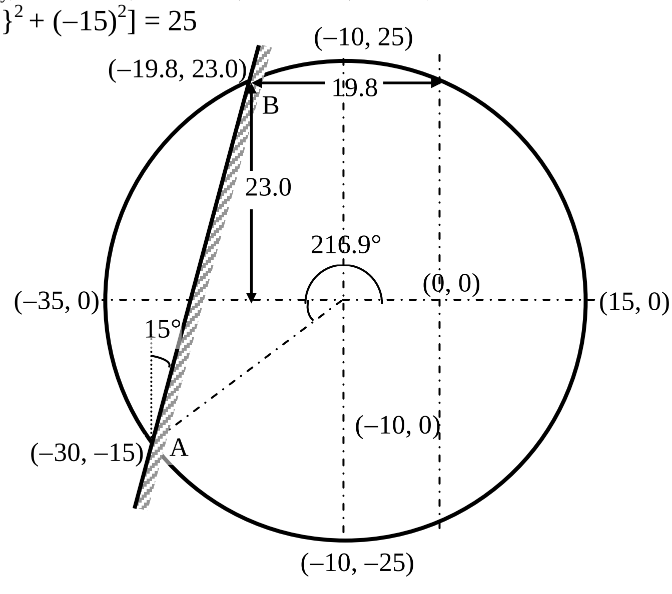
<u>Example</u>: For an infinitesimal element, $\sigma_{xx} = -30$ ksi, $\sigma_{yy} = 10$ ksi, and $\tau_{xy} = -15$ ksi. In Mohr's circle of stress, show the normal and shear stresses acting on a plane defined by $\theta = -15°$.

The coordinates of the center of the circle = $[(\sigma_{xx} + \sigma_{yy})/2, 0] = [(-30 + 10)/2, 0] = (-10, 0)$ and radius $R = \sqrt{[\{(\sigma_{xx}-\sigma_{yy})/2\}^2 + (\tau_{xy})^2]} = \sqrt{[\{(-30-10)/2\}^2 + (-15)^2]} = 25$

The principal stresses are
$\sigma_1 = a + R = -10 + 25 = 15$ ksi,
and $\sigma_2 = -10-25 = -35$ ksi.
$\tau_{max} = R = 25$ ksi, and $\tau_{min} = -25$ ksi.

$\tan \phi = 2\tau_{xy}/(\sigma_{xx}-\sigma_{yy}) = (-30)/(-40) \Rightarrow \phi = \alpha = 216.9°$

The coordinates of B = $(-19.8, 23.0) = (\sigma_{xx}', -\tau_{xy}')$
$\therefore \sigma_{xx}' = -19.8$ ksi, $\tau_{xy}' = -23.0$ ksi



Yield and Fracture Criteria

The studies on transformation of stress are aimed at developing the state of critical normal and shear stresses and the corresponding surfaces. But no comprehensive theory is available to predict the precise response of all types of real materials to such stresses, incorporating the multitude effects of static, dynamic, impact and cyclic loading, as well as temperature. Only the classical idealizations of yielding criteria for ductile materials and fracture criteria for brittle materials are discussed here, both of which are greatly affected by the temperature as well as the state of stress itself. All the criteria discussed here are formulated with respect to the principal stresses σ_1 and σ_2 and based on comparison with the yield criteria for materials under uniaxial tension, assuming identical material properties in tension and compression.

1. <u>Maximum Normal Stress Theory (Rankine)</u>:
   According to this theory, *yielding occurs when the maximum normal stress (σ_1 or σ_2) at a point equals the maximum normal stress at yield in the uniaxial tension test*.
   Since the maximum normal stress at yield in the uniaxial tension test = Y (the yield strength of the material), the yield criterion becomes $|\sigma_1| \geq Y$, or $|\sigma_2| \geq Y$.
   This can be plotted with σ_1 in the x-axis and σ_2 in the y-axis as shown in Fig. 1.

2. <u>Maximum Normal Strain Theory (St. Venant)</u>:
   According to this theory, *yielding occurs when the maximum normal strain (ε_1 or ε_2) at a point equals the maximum normal strain at yield in the uniaxial tension test*.
   Since $\varepsilon_1 = (\sigma_1 - \nu\sigma_2)/E$, $\varepsilon_2 = (\sigma_2 - \nu\sigma_1)/E$ [where E = Young's modulus, ν = Posson's ratio], and the maximum normal strain at yield in the uniaxial tension test = Y/E,
   the yield criterion becomes $|\sigma_1 - \nu\sigma_2| \geq Y$, or $|\sigma_2 - \nu\sigma_1| \geq Y$.
   This is plotted with σ_1 in the x-axis and σ_2 in the y-axis as shown in Fig. 2.

3. <u>Maximum Shear Stress Theory (Tresca)</u>:
   According to this theory, *yielding occurs when the maximum shear stress (τ_{max}) at a point equals the maximum shear stress at yield in the uniaxial tension test*.
   Since $\tau_{max} = |\sigma_1 - \sigma_2|/2$ and the maximum shear stress at yield in the uniaxial tension test = Y/2 (the yield strength of the material), the yield criterion becomes $|\sigma_1 - \sigma_2| \geq Y$, and adding the normal stress criterion $|\sigma_1| \geq Y$, or $|\sigma_2| \geq Y$, is plotted with σ_1 in the x-axis and σ_2 in the y-axis as shown in Fig. 3.

4. <u>Maximum Distortion-Energy Theory (Von Mises)</u>:
   According to this theory, *yielding occurs when the maximum distortion strain energy (U_{dist}) at a point equals the maximum distortion strain energy at yield in the uniaxial tension test*.
   Since U_{dist} for 3-dimensional stress is = $\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}/12G$, i.e., for 2-dimensional stress is = $\{(\sigma_1-\sigma_2)^2 + (\sigma_2-0)^2 + (0-\sigma_1)^2\}/12G = \{(\sigma_1-\sigma_2)^2 + \sigma_2^2 + \sigma_1^2\}/12G$, and U_{dist} at yield in the uniaxial tension test = $2Y^2/12G$, the yield criterion becomes $\{(\sigma_1-\sigma_2)^2 + \sigma_2^2 + \sigma_1^2\} = 2Y^2$; i.e.,
   $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = Y^2$, which is plotted with σ_1 in the x-axis and σ_2 in the y-axis in Fig. 4.

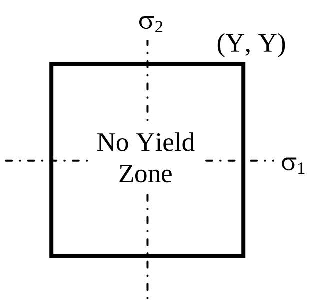


Fig. 1

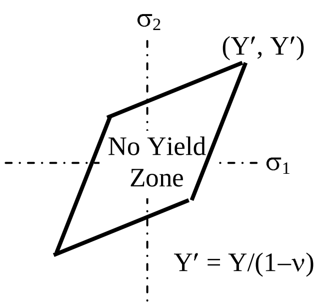


Fig. 2

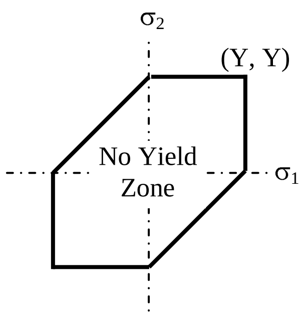


Fig. 3

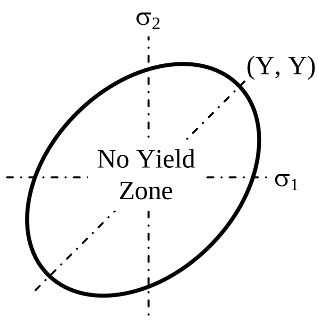
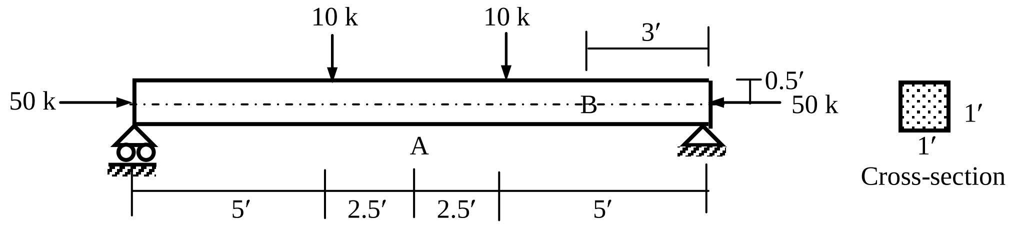


Fig. 4

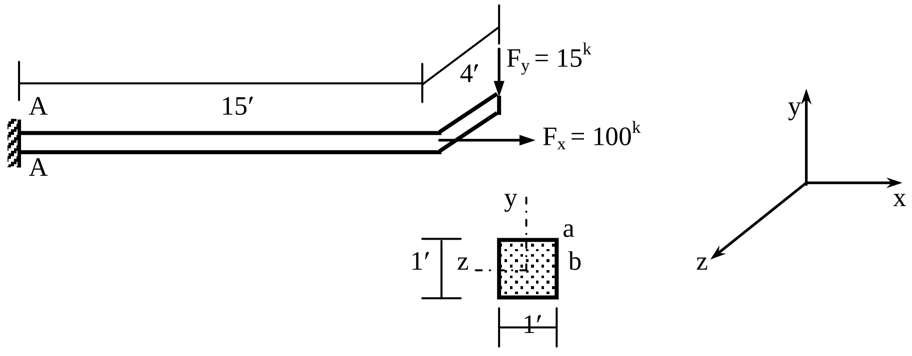
Practice Problems on Transformation of Stress

1. In the beam shown below, calculate the principal stresses and show the principal planes at A and B.



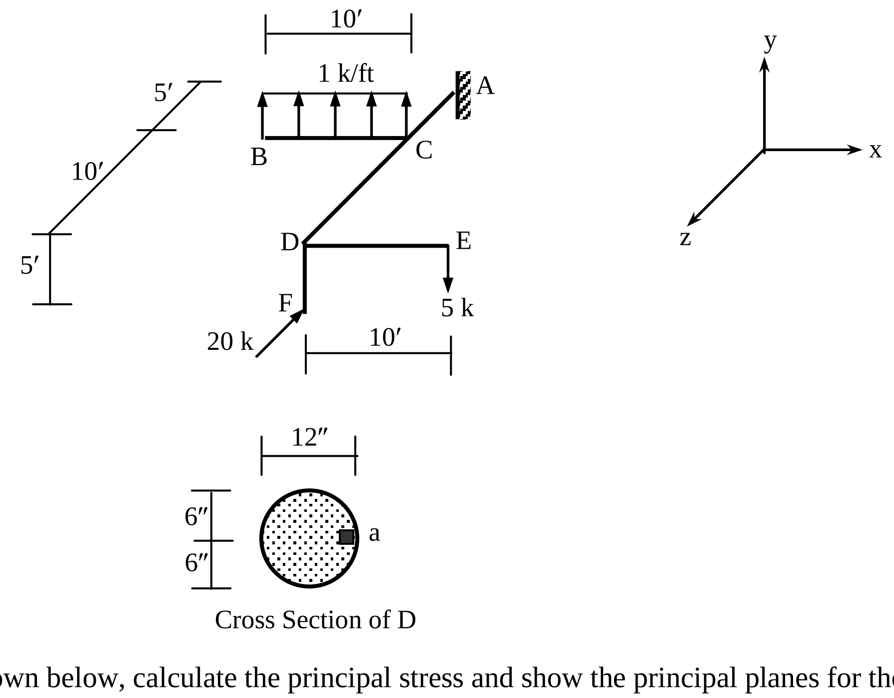
Side Elevation

2. For the structure shown below, calculate the stresses (σ_x, σ_y, τ_{xy}) and the principal stresses (σ_1, σ_2) at points a and b of section A-A.



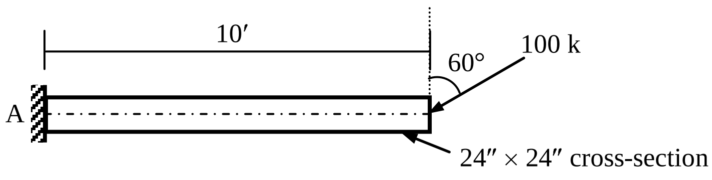
Section A-A

3. For the structure shown below, calculate the principal stresses (and corresponding angles) at point 'a' of the section D.

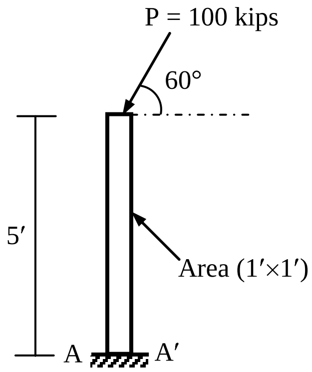


Cross Section of D

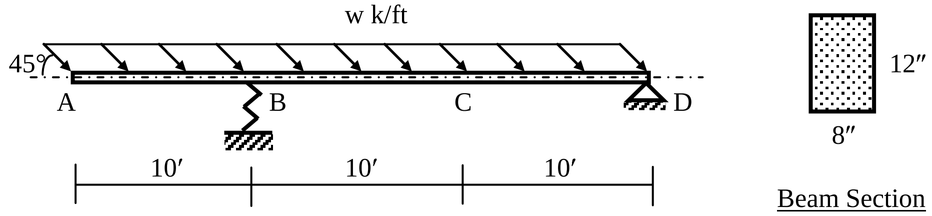
4. In the beam shown below, calculate the principal stress and show the principal planes for the point A.



5. In the column shown below, use transformation of stress to calculate the maximum shear stress at Section A-A′. Also indicate the point and surface where it occurs in the section.



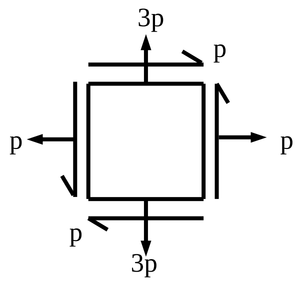
6. The shear stress at an element on the neutral axis of section D of beam ABCD shown below is 50 psi. For this element
   (i) calculate normal stresses (σ_{xx}, σ_{yy}), principal stresses (σ_1, σ_2) and maximum shear stress (τ_{max}),
   (ii) draw the Mohr's circle of stresses.



7. For an infinitesimal element, σ_x = –30 ksi, σ_y = 10 ksi, and τ_{xy} = –15 ksi. In Mohr's circle of stress, show the normal and shear stresses acting on a plane defined by θ = –45°.

8. The coordinate of the center of a Mohr's circle is (30, 0) and its radius is 12. If the principal plane is located at an angle θ = 30° from plane X-X, calculate the normal stresses (σ_x, σ_y) and shear stress (τ_{xy}) on that plane [all stresses are in ksi]. Also show these stresses graphically on the Mohr's circle.

9. For the stress condition in the element shown below, find the maximum allowable value of p using the Von Mises yielding criterion, if the yield strength of the material is 40 ksi.



10. Calculate the shear stress necessary to cause yielding of a material in a pure shear condition. Use the Von Mises yielding criterion, if the yield strength of the material is 36 ksi.

11. The coordinates of the center of a Mohr's circle is ($-p_0/2$, 0) and its radius is p_0.

    Calculate the maximum allowable value of p_0 if the material is to avoid yielding using all the criteria suggested below, by (i) Rankine, (ii) St. Venant, (iii) Tresca, (iv) Von Mises

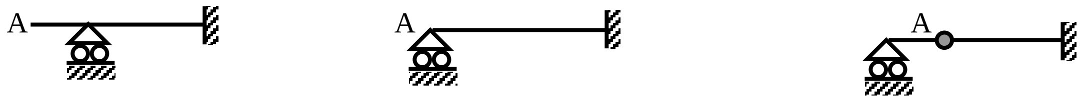
    [Given: Yield Strength of the material = 400 MPa, Poisson's ratio = 0.25].

12. For the stress condition described in Problem 7, calculate the required yield strength Y to avoid yielding of the material using the yield criteria suggested by
    (i) Rankine, (ii) St. Venant, (iii) Tresca, (iv) Von Mises [Given: Poisson's ratio = 0.25].

Bending Moment Diagram (BMD)

1. BM = 0 at points A

   (i) Free End (ii) Hinge/Roller Supported End (iii) Internal Hinge

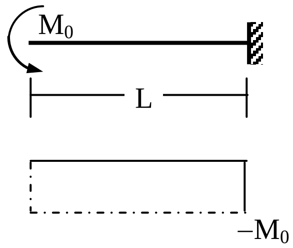


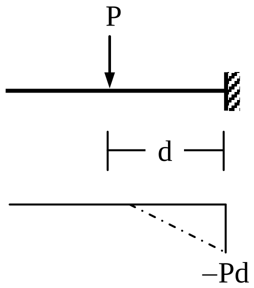
2. BM ≠ 0 at points B (in general, but can be = 0 only for special loading cases)

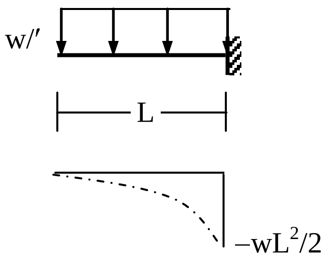
   (i) Fixed End (ii) Internal Roller/Hinge support

3. Verify and memorize the following BMDs

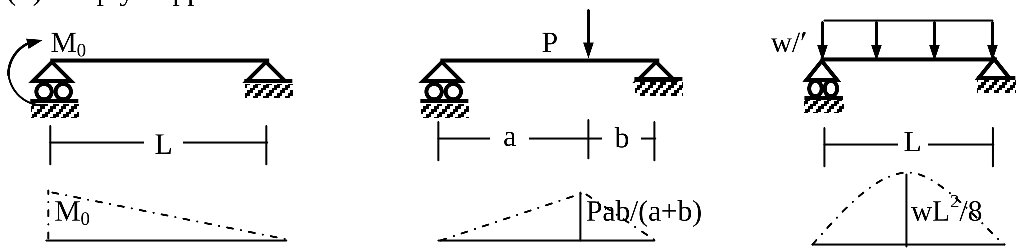
   (i) Cantilever Beams





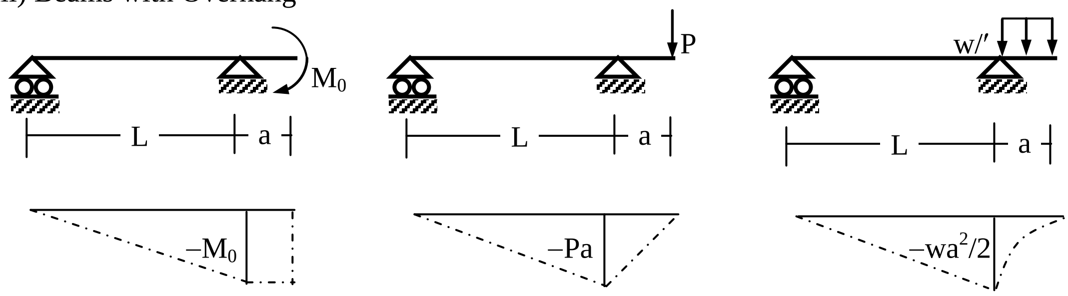


   (ii) Simply Supported Beams

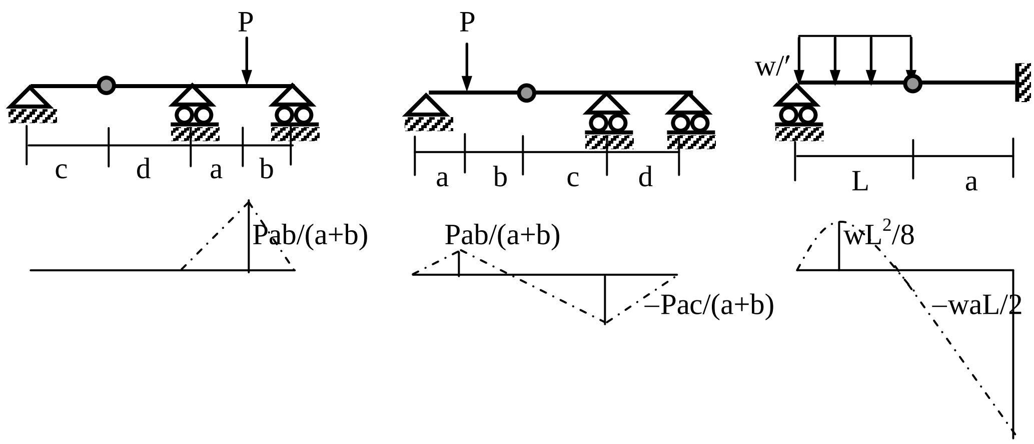


(= PL/4, if a = b = L/2)

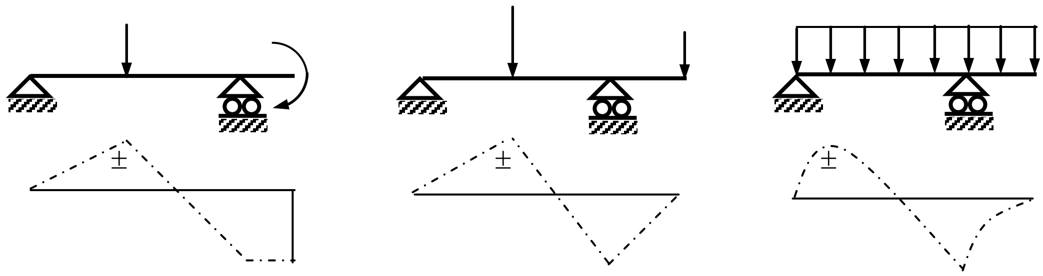
   (iii) Beams with Overhang



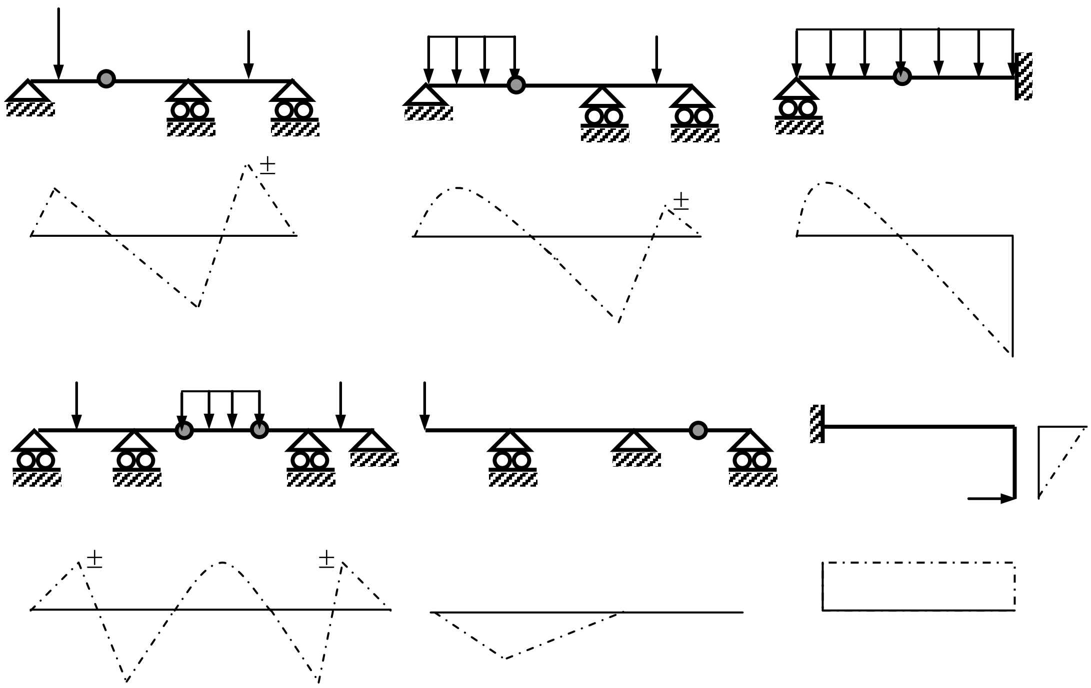
   (iv) Beams with Internal Hinge



4. Qualitative BMDs



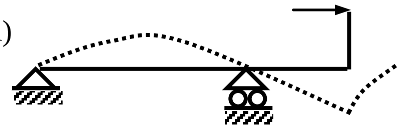
[± implies that the ordinate can be positive or negative depending on the loads and spans]



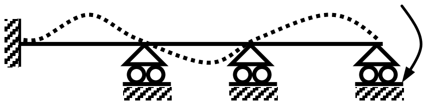
Qualitative Deflected Shapes

1. The following figures show qualitative deflected shapes for the beams due to the applied loads.

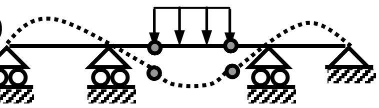
(i)



(ii)



(iii)



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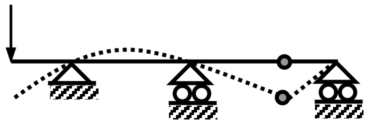
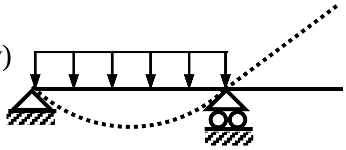


(v)



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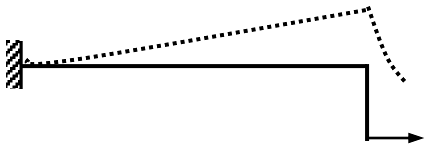
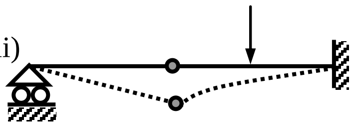
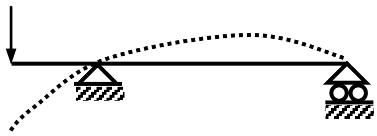


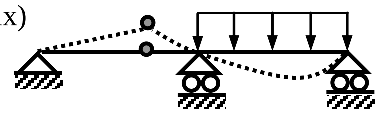
(vii)



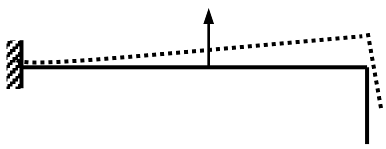
(viii)



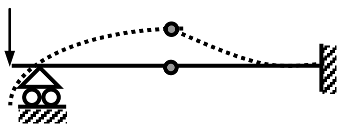
(ix)



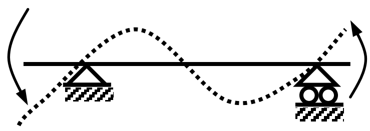
(x)



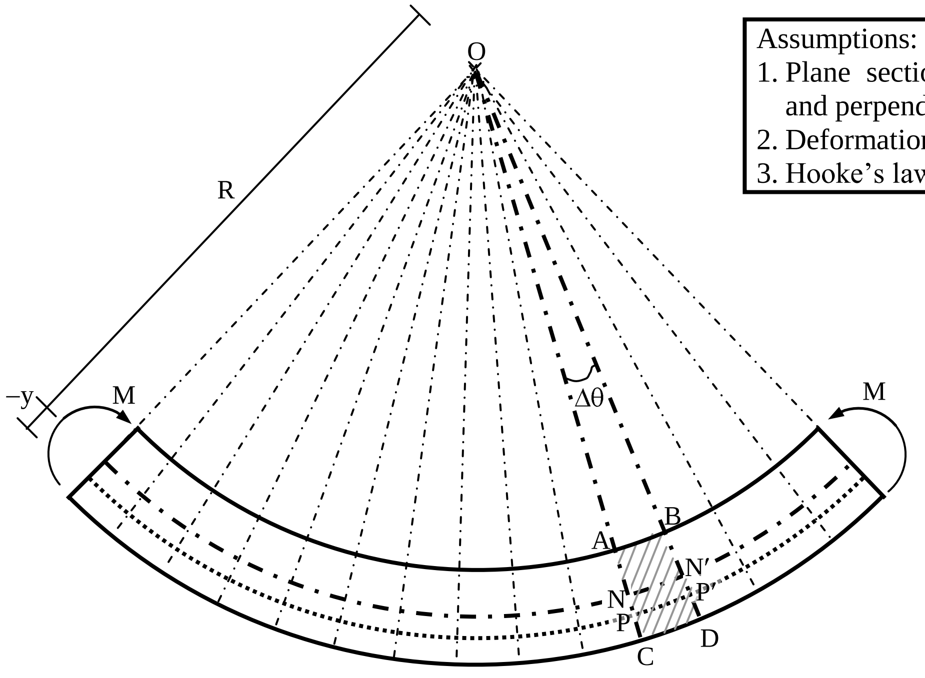
(xi)



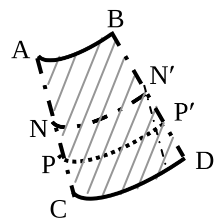
(xii)



Moment-Curvature Relationship



Assumptions:
1. Plane sections remain plane and perpendicular to NA
2. Deformations are small
3. Hooke's law is valid



If $NN' = \Delta s$ and $PP' = \Delta s + \Delta u$

$\therefore$ If $\Delta s \to 0$, Axial Strain $\varepsilon_x = \Delta u/\Delta s \to du/ds$ ……..(i)

Also, $\Delta u = -y\,\Delta\theta \Rightarrow \Delta u/\Delta s = -y\,\Delta\theta/\Delta s$ ………(ii)

$\therefore$ If $\Delta s \to 0$, $du/ds = -y\,d\theta/ds$ ………(iii)

Also, Curvature $\kappa = 1/R = d\theta/ds$

$\therefore$ Eq. (i) $\Rightarrow \varepsilon_x = -y\,\kappa$ ………(iv)

Using stress-strain relationship, $\varepsilon_x = \sigma_x/E$ and also $\sigma_x = -My/I \Rightarrow -My/EI = -y\,\kappa$

$\therefore$ Curvature, $\kappa = M/EI$ ………(v), the moment-curvature relationship

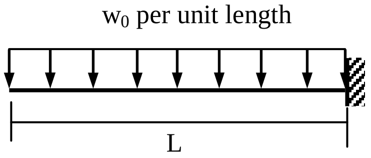
and Radius of Curvature, $R = 1/\kappa = EI/M$ ………(vi)

From analytic geometry, $\kappa = (d^2v/dx^2)/\{1+(dv/dx)^2\}^{3/2} \cong d^2v/dx^2$, if $dv/dx \to 0$

$\therefore$ (v) $\Rightarrow d^2v/dx^2 \cong M/EI$; i.e., $\Rightarrow$ Bending Moment, $M \cong EI\,d^2v/dx^2$ ……….(vii)

Also, Shear Force $V = dM/dx \cong EI\,d^3v/dx^3$ and Load $w = dV/dx \cong EI\,d^4v/dx^4$………(viii)

Example: Calculate the tip deflection of the cantilever beam shown below [Given: EI = const].



$EI\,d^4v/dx^4 \cong -w_0$

$\Rightarrow V(x) = -w_0\,x + C_1$ …………..(1)

$\Rightarrow M(x) = -w_0\,x^2/2 + C_1\,x + C_2$ …………..(2)

$V(0) = 0 \Rightarrow C_1 = 0$ [from (1)], $M(0) = 0 \Rightarrow C_2 = 0$ [from (2)]

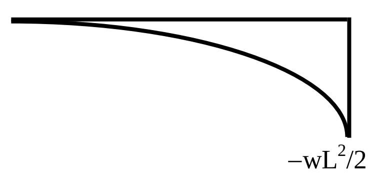
$\Rightarrow EIv'(x) \cong -w_0\,x^3/6 + C_3$ …………..(3)

and $EIv(x) \cong -w_0\,x^4/24 + C_3\,x + C_4$ …………..(4)

$v'(L) = 0 \Rightarrow C_3 = w_0\,L^3/6$ [from (3)]

$v(L) = 0 \Rightarrow C_4 = -w_0\,L^4/8$ [from (4)]

$\therefore$ Eq. (4) $\Rightarrow v(0) = C_4/EI = -w_0\,L^4/8EI$



Calculation of Deflection using Singularity Functions

Singularity Functions:

$f(x) = \langle x-a\rangle^n \Rightarrow f(x) = 0$, when $x \le a$; and $f(x) = \langle x-a\rangle^n$, when $x > a$ [where $n \ge 0$]

$\therefore f(x) = \langle x-a\rangle^0 \Rightarrow f(x) = 0$, when $x \le a$; and $f(x) = 1$, when $x > a$

However $\langle x-a\rangle^n$ has no physical significance if $n < 0$, and is written only as a notation with an asterisk ($_*$) as subscript; e.g., $f(x) = \langle x-a\rangle^{-1}{}_*$

The integration and differentiation of singularity functions follow the rules for ordinary polynomial functions; i.e., $\int\langle x-a\rangle^n\,dx = \langle x-a\rangle^{n+1}/(n+1) + C_1$ and $d(\langle x-a\rangle^n)/dx = n\langle x-a\rangle^{n-1}$

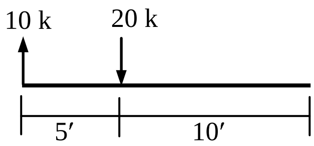
e.g., $\int\langle x-a\rangle^2\,dx = \langle x-a\rangle^3/3 + C_1$ and $d(\langle x-a\rangle^2)/dx = 2\langle x-a\rangle^1$

By definition, $\int\langle x-a\rangle^n{}_*\,dx = \langle x-a\rangle^{n+1}{}_* + C_1$ if $n < 0$;

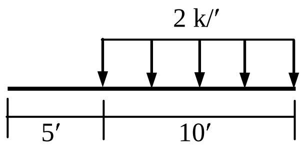
e.g., $\int\langle x-a\rangle^{-2}{}_*\,dx = \langle x-a\rangle^{-1}{}_* + C_1$ and $\int\langle x-a\rangle^{-1}{}_*\,dx = \langle x-a\rangle^0 + C_1$

Singularity Functions for Common Loadings:

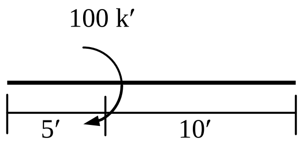
Common loadings are expressed in terms of the following singularity functions



$w(x) = 10\langle x-0\rangle^{-1}{}_* - 20\langle x-5\rangle^{-1}{}_*$

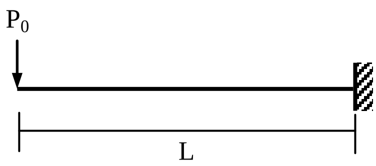


$w(x) = -2\langle x-5\rangle^0$



$w(x) = 100\langle x-5\rangle^{-2}{}_*$

Example 1: Calculate the tip deflection of the cantilever beam shown below [Given: EI = const].



$EI\, d^4v/dx^4 \cong w(x) = -P_0\langle x-0\rangle^{-1}{}_*$

$\Rightarrow V(x) = -P_0\langle x-0\rangle^0 + C_1$ …………..(1)

$\Rightarrow M(x) = -P_0\langle x-0\rangle^1 + C_1 x + C_2$ …………..(2)

$V(0) = 0 \Rightarrow C_1 = 0$ [from (1)], $M(0) = 0 \Rightarrow C_2 = 0$ [from (2)]

$\Rightarrow EIv'(x) \cong -P_0\langle x-0\rangle^2/2 + C_3$ …………..(3)

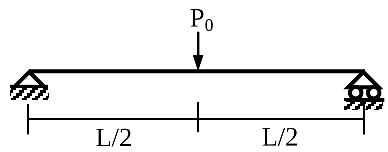
and $EIv(x) \cong -P_0\langle x-0\rangle^3/6 + C_3 x + C_4$ …………..(4)

$v'(L) = 0 \Rightarrow C_3 = P_0 L^2/2$ [from (3)]

$v(L) = 0 \Rightarrow C_4 = -P_0 L^3/3$ [from (4)]

$\therefore$ Eq. (4) $\Rightarrow v(0) = C_4/EI = -P_0 L^3/3EI$

Example 2: Calculate the midspan deflection of the simply supported beam shown below.



$EI\, d^4v/dx^4 \cong w(x) = -P_0\langle x-L/2\rangle^{-1}{}_*$

$\Rightarrow V(x) \cong -P_0\langle x-L/2\rangle^0 + C_1$ …………..(1)

$\Rightarrow M(x) \cong -P_0\langle x-L/2\rangle^1 + C_1 x + C_2$ …………..(2)

$M(0) = 0 \Rightarrow C_2 = 0$ [from (2)], $M(L) = 0 \Rightarrow C_1 = P_0/2$ [from (2)]

$\Rightarrow EIv'(x) \cong -P_0\langle x-L/2\rangle^2/2 + (P/2)\,x^2/2 + C_3$ …………..(3)

and $EIv(x) \cong -P_0\langle x-L/2\rangle^3/6 + (P/2)\,x^3/6 + C_3 x + C_4$ …………..(4)

$v(0) = 0 \Rightarrow C_4 = 0$ [from (4)]

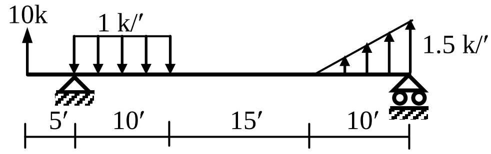
$v(L) = 0 \Rightarrow C_3 = -P_0 L^2/16$ [from (4)]

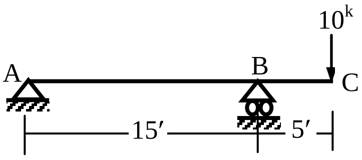
$\therefore$ Eq. (4) $\Rightarrow EI\,v(L/2) = (P/2)\,(L/2)^3/6 + (-P_0 L^2/16)\,L/2 \Rightarrow v(L/2) = -P_0 L^3/48EI$

Example 3: Derive equation of the deflected shape of the beam shown below.

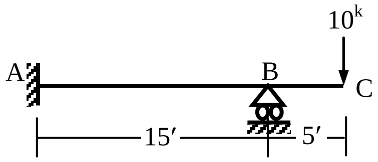
$w(x) = 10\langle x-0\rangle^{-1}{}_* + R_1\langle x-5\rangle^{-1}{}_* - 1\langle x-5\rangle^0 + 1\langle x-15\rangle^0 + 0.15\langle x-30\rangle^1$

BCs: $V(0) = 0$, $M(0) = 0$, $M(40) = 0$,
$v(5) = 0$, $v(40) = 0$

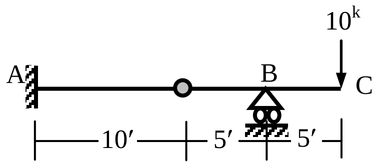




Beam1



Beam2



Beam3

1. <u>Statically Determinate Beam with Overhang</u>

$EIv^{iv}(x) \cong w(x) = R_B\langle x-15\rangle^{-1}_{*} - 10\langle x-20\rangle^{-1}_{*}$

$EIv'''(x) \cong V(x) = R_B\langle x-15\rangle^{0} - 10\langle x-20\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = R_B\langle x-15\rangle^{1} - 10\langle x-20\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong R_B/2\langle x-15\rangle^{2} - 5\langle x-20\rangle^{2} + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong R_B/6\langle x-15\rangle^{3} - 5/3\langle x-20\rangle^{3} + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0$ (5)

$D(0) = 0 \Rightarrow C_4 = 0$ (6)

$V(20) = 0 \Rightarrow R_B - 10 + C_1 = 0$ (7)

$M(20) = 0 \Rightarrow 5R_B + 20\,C_1 = 0$ (8)

$D(15) = 0 \Rightarrow 562.5\,C_1 + 15\,C_3 = 0$ (9)

Solving (7), (8), (9) $\Rightarrow C_1 = -3.33,\ C_3 = 125,\ R_B = 13.33$ (10)

$\therefore EIv(x) = D(x) \cong 2.222\langle x-15\rangle^{3} - 1.667\langle x-20\rangle^{3} - 0.555\,x^3 + 125\,x$ (11)

2. <u>Statically Indeterminate Beam with Overhang</u>

$EIv^{iv}(x) \cong w(x) = R_B\langle x-15\rangle^{-1}_{*} - 10\langle x-20\rangle^{-1}_{*}$

$EIv'''(x) \cong V(x) = R_B\langle x-15\rangle^{0} - 10\langle x-20\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = R_B\langle x-15\rangle^{1} - 10\langle x-20\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong R_B/2\langle x-15\rangle^{2} - 5\langle x-20\rangle^{2} + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong R_B/6\langle x-15\rangle^{3} - 5/3\langle x-20\rangle^{3} + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$S(0) = 0 \Rightarrow C_3 = 0$ (5)

$D(0) = 0 \Rightarrow C_4 = 0$ (6)

$V(20) = 0 \Rightarrow R_B - 10 + C_1 = 0$ (7)

$M(20) = 0 \Rightarrow 5R_B + 20\,C_1 + C_2 = 0$ (8)

$D(15) = 0 \Rightarrow 562.5\,C_1 + 112.5\,C_2 = 0$ (9)

Solving (7), (8), (9) $\Rightarrow C_1 = -5,\ C_2 = 25,\ R_B = 15$ (10)

$\therefore EIv(x) \cong D(x) = 2.5\langle x-15\rangle^{3} - 1.667\langle x-20\rangle^{3} - 0.833\,x^3 + 12.5\,x^2$ (11)

3. <u>Statically Determinate Beam with Overhang & Internal Hinge</u>

$EIv^{iv}(x) \cong w(x) = C_\theta\langle x-10\rangle^{-3}_{*} + R_B\langle x-15\rangle^{-1}_{*} - 10\langle x-20\rangle^{-1}_{*}$

$EIv'''(x) \cong V(x) = C_\theta\langle x-10\rangle^{-2}_{*} + R_B\langle x-15\rangle^{0} - 10\langle x-20\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = C_\theta\langle x-10\rangle^{-1}_{*} + R_B\langle x-15\rangle^{1} - 10\langle x-20\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong C_\theta\langle x-10\rangle^{0} + R_B/2\langle x-15\rangle^{2} - 5\langle x-20\rangle^{2} + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong C_\theta\langle x-10\rangle^{1} + R_B/6\langle x-15\rangle^{3} - 5/3\langle x-20\rangle^{3} + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$S(0) = 0 \Rightarrow C_3 = 0$ (5)

$D(0) = 0 \Rightarrow C_4 = 0$ (6)

$V(20) = 0 \Rightarrow R_B - 10 + C_1 = 0$ (7)

$M(10) = 0 \Rightarrow 10\,C_1 + C_2 = 0$ (8)

$M(20) = 0 \Rightarrow 5R_B + 20\,C_1 + C_2 = 0$ (9)

$D(15) = 0 \Rightarrow 562.5\,C_1 + 112.5\,C_2 + 5\,C_\theta = 0$ (10)

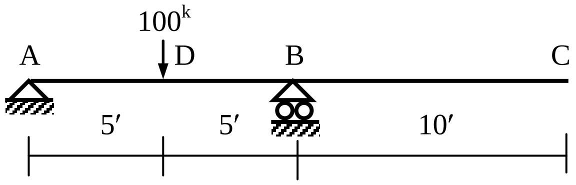
Solving (7), (8), (9) $\Rightarrow C_1 = -10,\ C_2 = 100,\ R_B = 20;\ \therefore$ (10) $\Rightarrow C_\theta = -1125$ (11)

$\therefore EIv(x) = D(x) \cong -1125\langle x-10\rangle^{1} + 3.333\langle x-15\rangle^{3} - 1.667\langle x-20\rangle^{3} - 1.667\,x^3 + 50\,x^2$ (12)

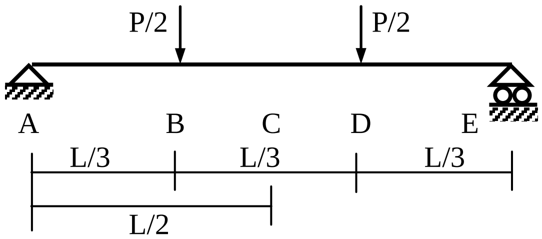
Practice Problems on Beam Deflection

2. Calculate the deflection and rotation at point B [EI = constant].

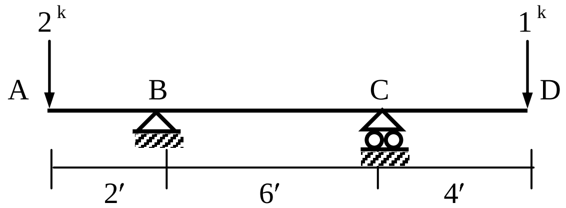
3. Calculate the deflection and rotation at point C [EI = 40,000 k-ft^2].



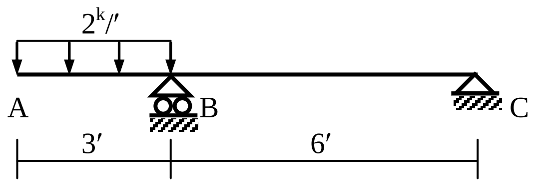
4. Calculate the deflection at C and the rotation at A [EI = constant].



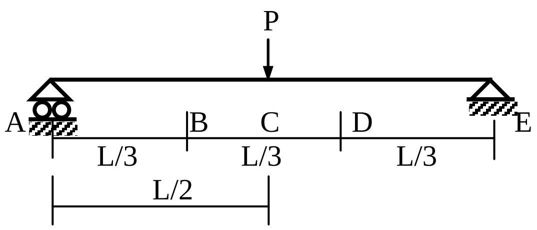
5. Calculate the deflection at A and the rotation at B [EI = 40,000 k-ft^2].



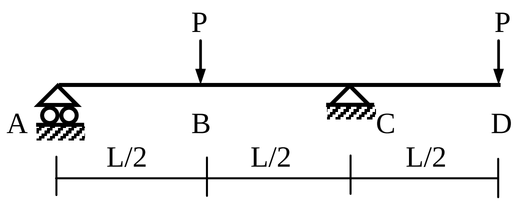
6. Calculate the deflection at A [EI_{AB} = 40,000 k-ft^2, EI_{BC} = 20,000 k-ft^2].



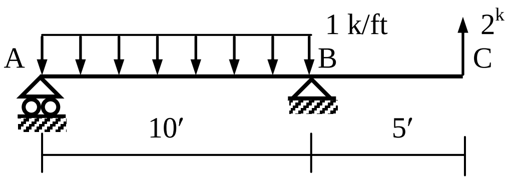
7. Calculate the deflection at C [$EI_{AB} = EI_{DE} = EI$, $EI_{BCD} = 2EI$].



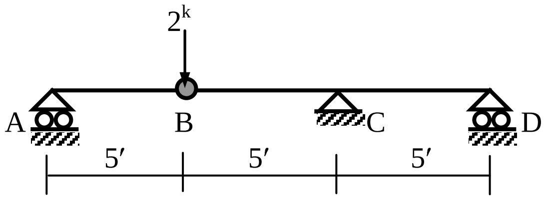
8. Calculate the deflection at D [EI = constant].



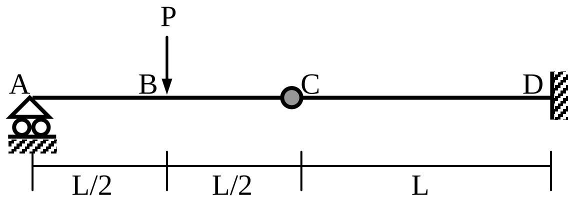
9. Calculate the deflection at C and the rotation at B [EI= 40,000 k-ft^2].



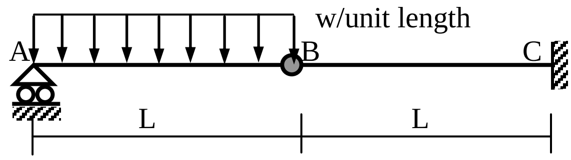
10. Calculate the deflection at B and rotations at the left and right of B [EI = 40,000 k-ft^2].



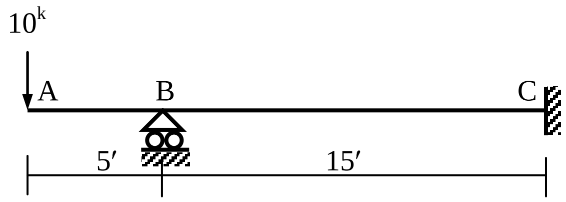
11. Calculate the deflection at C [EI = constant].



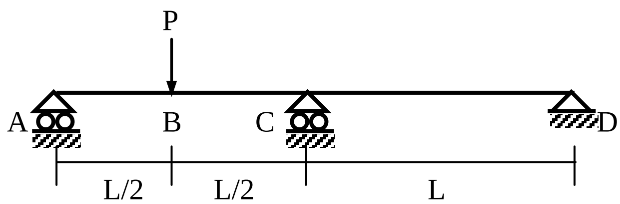
12. Calculate the deflection at B and rotations at the left and right of B [EI = constant].



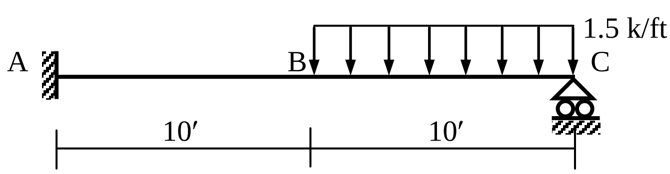
13. Calculate the reaction at support B [EI = constant].



14. Calculate the deflection at B [EI = constant].

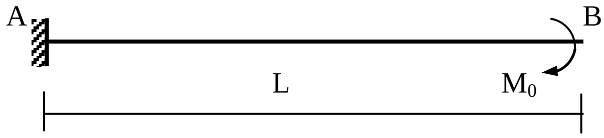


15. Calculate the deflection at B and the rotation at C [EI= 40,000 k-ft^2].



Beam Deflection Solutions using Singularity Functions

2. Calculate the deflection and rotation at point B [EI = constant].



$EIv^{iv}(x) \cong w(x) = M_0\langle x-L\rangle^{-2}{}_*$

$EIv'''(x) \cong V(x) = M_0\langle x-L\rangle^{-1}{}_* + C_1$ (1)

$EIv''(x) \cong M(x) = M_0\langle x-L\rangle^{0} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong M_0\langle x-L\rangle^{1} + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong M_0\langle x-L\rangle^{2}/2 + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$S(0) = 0 \Rightarrow C_3 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ V(L) = 0 \Rightarrow C_1 = 0$ (5~7)

$M(L) = 0 \Rightarrow M_0 + C_1 L + C_2 = 0 \Rightarrow C_2 = -M_0$ (8)

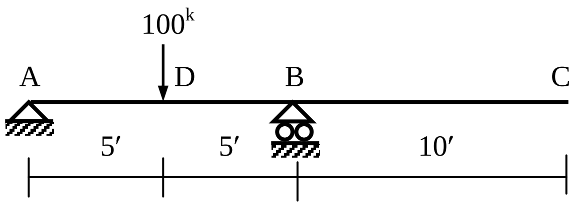
$\therefore EIv(x) = D(x) \cong M_0\langle x-L\rangle^{2}/2 - M_0 x^2/2$ (9)

$\therefore EIv'(x) = S(x) \cong M_0\langle x-L\rangle^{1} - M_0 x$ (10)

$\therefore v(L) \cong -M_0 L^2/2EI;\ v'(L) \cong -M_0 L/EI$ (11~12)

[Instead of $V(L) = 0$ and $M(L) = 0$,
one can use $\sum F_y = 0 \Rightarrow R_A = C_1 = 0;\ \sum M_A = 0 \Rightarrow C_2 + M_0 = 0 \Rightarrow C_2 = -M_0$]

3. Calculate the deflection and rotation at point C [EI = 40,000 k-ft^2].



$EIv^{iv}(x) \cong w(x) = -100\langle x-5\rangle^{-1}{}_* + R_B\langle x-10\rangle^{-1}{}_*$

$EIv'''(x) \cong V(x) = -100\langle x-5\rangle^{0} + R_B\langle x-10\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -100\langle x-5\rangle^{1} + R_B\langle x-10\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong -50\langle x-5\rangle^{2} + R_B\langle x-10\rangle^{2}/2 + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong -50\langle x-5\rangle^{3}/3 + R_B\langle x-10\rangle^{3}/6 + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ V(20) = -100 + R_B + C_1 = 0$ (5~7)

$M(20) = 0 \Rightarrow -1500 + 10\, R_B + 20\, C_1 = 0$ (8)

$D(10) = 0 \Rightarrow -\,2083.33 + 166.67\, C_1 + 10\, C_3 = 0$ (9)

Solving (7), (8) $\Rightarrow C_1 = 50,\ R_B = 50$ (10)

$\therefore$ (9) $\Rightarrow C_3 = -\,625$ (11)

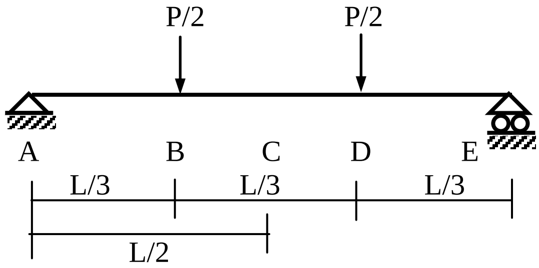
$EIv(x) = D(x) \cong -16.67\langle x-5\rangle^{3} + 8.33\langle x-10\rangle^{3} + 8.33\, x^3 - 625\, x$ (12)

$EIv'(x) = S(x) \cong -50\langle x-5\rangle^{2} + 25\langle x-10\rangle^{2} + 25\, x^2 - 625$ (13)

$v(20) \cong 6250/40{,}000 = 0.156$ ft, $v'(20) \cong 625/40{,}000 = 0.0156$ rad (14~15)

[Instead of $M(0) = 0$, $V(20) = 0$ and $M(20) = 0$,
one can use $\sum M_A = 0 \Rightarrow 100 \times 5 - 10\, R_B = 0 \Rightarrow R_B = 50$; A is hinged end $\Rightarrow C_2 = 0$;
$\sum F_y = 0 \Rightarrow C_1 + R_B = 100 \Rightarrow C_1 = 100 - 50 = 50$]

4. Calculate the deflection at C and the rotation at A [EI = constant].



$EIv^{iv}(x) \cong w(x) = -P/2<x-L/3>^{-1}{}_{*} - P/2<x-2L/3>^{-1}{}_{*}$

$EIv'''(x) \cong V(x) = -P/2<x-L/3>^{0} - P/2<x-2L/3>^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -P/2<x-L/3>^{1} - P/2<x-2L/3>^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong -P/2<x-L/3>^{2}/2 - P/2<x-2L/3>^{2}/2 + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong -P/2<x-L/3>^{3}/6 - P/2<x-2L/3>^{3}/6 + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ M(L) = 0 \Rightarrow C_1 = P/2$ (5~7)

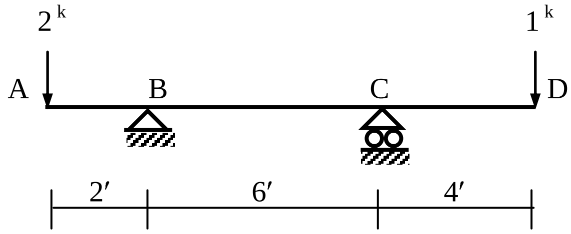
$D(L) = -8\,PL^3/324 - PL^3/324 + PL^3/12 + C_3 L = 0 \Rightarrow C_3 = -PL^2/18$ (8)

$EIv(x) = D(x) \cong -P/2<x-L/3>^{3}/6 - P/2<x-2L/3>^{3}/6 + Px^3/12 - PL^2x/18$ (9)

$D(L/2) \cong -P/2\,(L^3/216)/6 + PL^3/96 - PL^3/36 \Rightarrow v(L/2) \cong -23PL^3/1296EI$

[Instead of $M(0) = 0$ and $M(L) = 0$,
one can use $\sum M_E = 0 \Rightarrow C_1 L - (P/2)\,2L/3 - (P/2)\,L/3 = 0 \Rightarrow C_1 = P/2$; A is hinged end $\Rightarrow C_2 = 0$]

5. Calculate the deflection at A and the rotation at B [EI = 40,000 k-ft^2].



$EIv^{iv}(x) \cong w(x) = -2<x-0>^{-1}{}_{*} + R_B<x-2>^{-1}{}_{*} + R_C<x-8>^{-1}{}_{*} - 1<x-12>^{-1}{}_{*}$

$EIv'''(x) \cong V(x) = -2<x-0>^{0} + R_B<x-2>^{0} + R_C<x-8>^{0} - 1<x-12>^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -2<x-0>^{1} + R_B<x-2>^{1} + R_C<x-8>^{1} - 1<x-12>^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong -<x-0>^{2} + R_B<x-2>^{2}/2 + R_C<x-8>^{2}/2 - 1<x-12>^{2}/2 + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong -<x-0>^{3}/3 + R_B<x-2>^{3}/6 + R_C<x-8>^{3}/6 - 1<x-12>^{3}/6 + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$V(0) = 0 \Rightarrow C_1 = 0;\ M(0) = 0 \Rightarrow C_2 = 0;\ V(12) = -2 + R_B + R_C - 1 = 0 \Rightarrow R_B + R_C = 3$ (5~7)

$M(12) = -24 + 10\,R_B + 4\,R_C = 0 \Rightarrow 10\,R_B + 4\,R_C = 24$ (8)

Solving (7), (8) $\Rightarrow R_B = 2,\ R_C = 1$ (9)

$D(2) = -8/3 + 2C_3 + C_4 = 0;\ D(8) = -512/3 + 36R_B + 8C_3 + C_4 = 0 \Rightarrow 8C_3 + C_4 = 296/3$ (10~11)

Solving (10), (11) $\Rightarrow C_3 = 16,\ C_4 = -88/3$ (12)

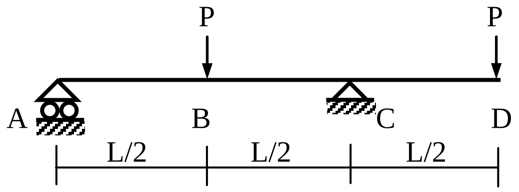
$EIv'(x) = S(x) \cong -<x-0>^{2} + <x-2>^{2} + 1<x-8>^{2}/2 - 1<x-12>^{2}/2 + 16$ (13)

$v(0) \cong C_4/40000 = -7.33 \times 10^{-4}$ ft (14)

$v'(2) \cong (-4 + 16)/40000 = -3 \times 10^{-4}$ rad (15)

[Instead of $V(0) = 0$, $M(0) = 0$, one can use A is free end $\Rightarrow V_A = C_1 = 0$ and $M_A = C_2 = 0$.
Also $\sum M_C = 0 \Rightarrow -2 \times 8 + R_B \times 6 + 1 \times 4 = 0 \Rightarrow R_B = 2;\ \sum F_y = 0 \Rightarrow R_B + R_C = 3 \Rightarrow R_C = 3 - 2 = 1$]

8. Calculate the deflection at D [EI = constant].



$EIv^{iv}(x) \cong w(x) = -P\langle x-L/2\rangle^{-1}_{*} + R_C\langle x-L\rangle^{-1}_{*} - P\langle x-3L/2\rangle^{-1}_{*}$

$EIv'''(x) \cong V(x) = -P\langle x-L/2\rangle^{0} + R_C\langle x-L\rangle^{0} - P\langle x-3L/2\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -P\langle x-L/2\rangle^{1} + R_C\langle x-L\rangle^{1} - P\langle x-3L/2\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong -P\langle x-L/2\rangle^{2}/2 + R_C\langle x-L\rangle^{2}/2 - P\langle x-3L/2\rangle^{2}/2 + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x) = D(x) \cong -P\langle x-L/2\rangle^{3}/6 + R_C\langle x-L\rangle^{3}/6 - P\langle x-3L/2\rangle^{3}/6 + C_1x^3/6 + C_2x^2/2 + C_3 x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0$; $D(0) = 0 \Rightarrow C_4 = 0$; $V(3L/2) = 0 \Rightarrow R_C + C_1 = 2P$ (5~7)

$M(3L/2) = 0 \Rightarrow -PL + R_C L/2 + (3L/2)C_1 = 0$ (8)

Solving (7), (8) $\Rightarrow R_C = 2P,\ C_1 = 0$ (9)

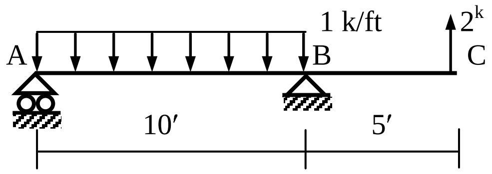
$D(L) = -PL^3/48 + C_3 L = 0 \Rightarrow C_3 = PL^2/48$ (10)

$EIv(x) = D(x) \cong -P\langle x-L/2\rangle^{3}/6 + P\langle x-L\rangle^{3}/3 - P\langle x-3L/2\rangle^{3}/6 + PL^2x/48$ (11)

$v(3L/2) \cong (-PL^3/6 + PL^3/24 + PL^3/32)/EI = -3PL^3/32\,EI$ (12)

[Instead of M(0) = 0, V(3L/2) = 0 and M(3L/2) = 0, one can use
A is hinged end $\Rightarrow C_2 = 0$; $\sum M_C = 0 \Rightarrow C_1 L + (P) L/2 - (P) L/2 = 0 \Rightarrow C_1 = 0$;
$\sum F_y = 0 \Rightarrow C_1 + R_C = 2P \Rightarrow R_C = 2P$]

9. Calculate the deflection at C and the rotation at B [EI = 40,000 k-ft^2].



$EIv^{iv}(x) \cong w(x) = -1\langle x-0\rangle^{0} + 1\langle x-10\rangle^{0} + R_B\langle x-10\rangle^{-1}_{*} + 2\langle x-15\rangle^{-1}_{*}$

$EIv'''(x) \cong V(x) = -1\langle x-0\rangle^{1} + 1\langle x-10\rangle^{1} + R_B\langle x-10\rangle^{0} + 2\langle x-15\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -\langle x-0\rangle^{2}/2 + \langle x-10\rangle^{2}/2 + R_B\langle x-10\rangle^{1} + 2\langle x-15\rangle^{1} + C_1 x + C_2$ (2)

$EIv'(x) = S(x) \cong -\langle x-0\rangle^{3}/6 + \langle x-10\rangle^{3}/6 + R_B\langle x-10\rangle^{2}/2 + \langle x-15\rangle^{2} + C_1x^2/2 + C_2 x + C_3$ (3)

$EIv(x)=D(x) \cong -\langle x-0\rangle^{4}/24+\langle x-10\rangle^{4}/24+R_B\langle x-10\rangle^{3}/6+\langle x-15\rangle^{3}/3+C_1x^3/6+C_2x^2/2+C_3 x+ C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0$; $D(0) = 0 \Rightarrow C_4 = 0$; $V(15) = -15 + 5 + R_B + C_1 + 2 = 0 \Rightarrow R_B + C_1 = 8$ (5~7)

$M(15) = -112.5 + 12.5 + 5R_B + 15C_1 = 0 \Rightarrow 5R_B + 15C_1 = 100$ (8)

Solving (7), (8) $\Rightarrow R_B = 2,\ C_1 = 6$ (9)

$D(10) = -10^4/24 + 6 \times 10^3/6 + 10C_3 = 0 \Rightarrow C_3 = -58.33$ (10)

$EIv'(x) = S(x) \cong -x^3/6 + \langle x-10\rangle^{3}/6 + \langle x-10\rangle^{2} + \langle x-15\rangle^{2} + 3x^2 - 58.33$ (11)

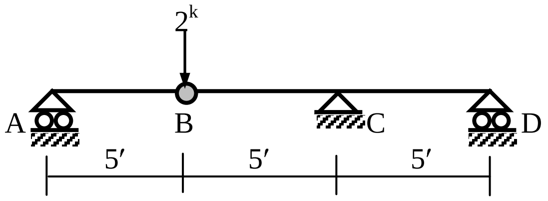
$EIv(x) = D(x) \cong -x^4/24 + \langle x-10\rangle^{4}/24 + \langle x-10\rangle^{3}/3 + \langle x-15\rangle^{3}/3 + x^3 - 58.33x$ (12)

$v'(10) \cong (-1000/6 + 300 - 58.33)/40000 = 1.875\times10^{-3}$ rad (13)

$v(15) \cong (-50625/24 + 625/24 + 125/3 + 3375 - 875)/40000 = 11.46\times10^{-3}$ ft (14)

[Instead of M(0) = 0, V(15) = 0 and M(15) = 0, one can use
A is hinged end $\Rightarrow C_2 = 0$; $\sum M_B = 0 \Rightarrow 10C_1 - (1\times10)5 - 2\times5 = 0 \Rightarrow C_1 = 6$;
$\sum F_y = 0 \Rightarrow C_1 + R_B - 10 + 2 = 0 \Rightarrow R_B = 2$]

10. Calculate the deflection at B and rotations at the left and right of B [EI = 40,000 k-ft^2].



$EIv^{iv}(x) \cong w(x) = -2\langle x-5\rangle^{-1}{}_* + C_\theta\langle x-5\rangle^{-3}{}_* + R_C\langle x-10\rangle^{-1}{}_*$

$EIv'''(x) \cong V(x) = -2\langle x-5\rangle^0 + C_\theta\langle x-5\rangle^{-2}{}_* + R_C\langle x-10\rangle^0 + C_1$ (1)

$EIv''(x) \cong M(x) = -2\langle x-5\rangle^1 + C_\theta\langle x-5\rangle^{-1}{}_* + R_C\langle x-10\rangle^1 + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -\langle x-5\rangle^2 + C_\theta\langle x-5\rangle^0 + R_C\langle x-10\rangle^2/2 + C_1x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -\langle x-5\rangle^3/3 + C_\theta\langle x-5\rangle^1 + R_C\langle x-10\rangle^3/6 + C_1x^3/6 + C_2x^2/2 + C_3\,x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ M(5) = 0 + 5\,C_1 = 0 \Rightarrow C_1 = 0$ (5~7)

$M(15) = -20 + 5\,R_C = 0 \Rightarrow R_C = 4$ (8)

$D(10) = -\,41.67 + 5\,C_\theta + 10\,C_3 = 0;\ D(15) = -333.33 + 10\,C_\theta + 83.33 + 15\,C_3 = 0$ (9)

Solving (8), (9) $\Rightarrow C_3 = -33.33,\ C_\theta = 75$ (10)

$EIv'(x) = S(x) \cong -\langle x-5\rangle^2 + 75\langle x-5\rangle^0 + 2\langle x-10\rangle^2 - 33.33$ (11)

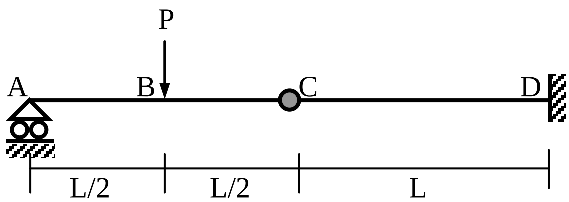
$EIv(x) = D(x) \cong -\langle x-5\rangle^3/3 + 75\langle x-5\rangle^1 + 2\langle x-10\rangle^3/3 - 33.33\,x$ (12)

$v'(5_-) \cong (-33.33)/40000 = -\,8.33\times10^{-4}$ rad; $v'(5_+) \cong (-33.33 + 75)/40000 = 1.04\times10^{-3}$ rad (13)

$v(5) \cong (-166.67)/40000 = -\,4.17\times10^{-3}$ ft (14)

[Instead of $M(0) = 0$, $M(5) = 0$ and $M(15) = 0$, one can use
A is hinged end $\Rightarrow C_2 = 0$; $\sum BM_B = 0 \Rightarrow C_1 \times 5 = 0 \Rightarrow C_1 = 0$;
$\sum M_D = 0 \Rightarrow R_C \times 5 - 2 \times 10 - C_1 \times 15 = 0 \Rightarrow R_C = 4$]

11. Calculate the deflection at C [EI = constant].



$EIv^{iv}(x) \cong w(x) = -P\langle x-L/2\rangle^{-1}{}_* + C_\theta\langle x-L\rangle^{-3}{}_*$

$EIv'''(x) \cong V(x) = -P\langle x-L/2\rangle^0 + C_\theta\langle x-L\rangle^{-2}{}_* + C_1$ (1)

$EIv''(x) \cong M(x) = -P\langle x-L/2\rangle^1 + C_\theta\langle x-L\rangle^{-1}{}_* + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -P\langle x-L/2\rangle^2/2 + C_\theta\langle x-L\rangle^0 + C_1x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -P\langle x-L/2\rangle^3/6 + C_\theta\langle x-L\rangle^1 + C_1x^3/6 + C_2x^2/2 + C_3\,x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ M(L) = -\,PL/2 + C_1\,L = 0 \Rightarrow C_1 = P/2$ (5~7)

$S(2L) = -9PL^2/8 + C_\theta + PL^2 + C_3 = 0 \Rightarrow C_\theta + C_3 = PL^2/8$ (8)

$D(2L) = -9PL^3/16 + C_\theta\,L + 2\,PL^3/3 + 2C_3L = 0 \Rightarrow C_\theta + 2\,C_3 = -5PL^2/48$ (9)

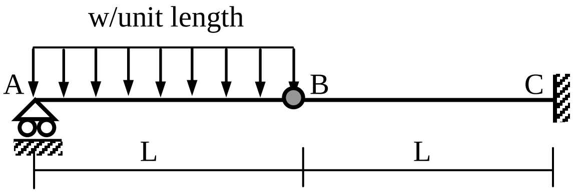
Solving (8), (9) $\Rightarrow C_3 = -11\,PL^2/48,\ C_\theta = 17\,PL^2/48$ (10)

$EIv(x) = D(x) \cong -P\langle x-L/2\rangle^3/6 + 17\,PL^2\langle x-L\rangle^1/48 + Px^3/12 - 11\,PL^2x/48$ (11)

$v(L) \cong (-\,PL^3/48 + PL^3/12 - 11\,PL^3/48)/EI = -\,PL^3/6EI$ (12)

[Instead of $M(0) = 0$ and $M(L) = 0$, one can use
A is hinged end $\Rightarrow C_2 = 0$; $\sum BM_C = 0 \Rightarrow C_1 \times L - P\times L/2 = 0 \Rightarrow C_1 = P/2$]

12. Calculate the deflection at B and rotations at the left and right of B [EI = constant].



$EIv^{iv}(x) \cong w(x) = -\,w\langle x-0\rangle^{0} + w\langle x-L\rangle^{0} + C_{\theta}\langle x-L\rangle^{-3}{}_{*}$

$EIv'''(x) \cong V(x) = -\,w\langle x-0\rangle^{1} + w\langle x-L\rangle^{1} + C_{\theta}\langle x-L\rangle^{-2}{}_{*} + C_1$ (1)

$EIv''(x) \cong M(x) = -\,w\langle x-0\rangle^{2}/2 + w\langle x-L\rangle^{2}/2 + C_{\theta}\langle x-L\rangle^{-1}{}_{*} + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -\,w\langle x-0\rangle^{3}/6 + w\langle x-L\rangle^{3}/6 + C_{\theta}\langle x-L\rangle^{0} + C_1\,x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -\,w\langle x-0\rangle^{4}/24 + w\langle x-L\rangle^{4}/24 + C_{\theta}\langle x-L\rangle^{1} + C_1\,x^3/6 + C_2\,x^2/2 + C_3\,x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0;\ M(L) = -\,wL^2/2 + C_1\,L = 0 \Rightarrow C_1 = wL/2$ (5~7)

$S(2L) = -8\,wL^3/6 + wL^3/6 + C_{\theta} + wL^3 + C_3 = 0 \Rightarrow C_{\theta} + C_3 = wL^3/6$ (8)

$D(2L) = -16\,wL^4/24 + wL^4/24 + C_{\theta}\,L + 8\,wL^4/12 + 2C_3L = 0 \Rightarrow C_{\theta} + 2\,C_3 = -wL^3/24$ (9)

Solving (8), (9) $\Rightarrow C_3 = -\,5\,wL^3/24,\ C_{\theta} = 3\,wL^3/8$ (10)

$EIv'(x) = S(x) \cong -\,wx^3/6 + w\langle x-L\rangle^{3}/6 + 3\,wL^3\langle x-L\rangle^{0}/8 + wL\,x^2/4 - 5wL^3/24$ (11)

$EIv(x) = D(x) \cong -\,wx^4/24 + w\langle x-L\rangle^{4}/24 + 3\,wL^3\langle x-L\rangle^{1}/8 + wL\,x^3/12 - 5wL^3x/24$ (12)

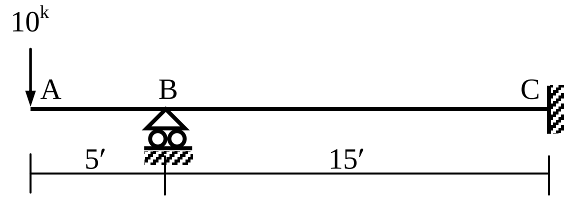
$v'(L_-) \cong (-\,wL^3/6 + wL^3/4 - 5wL^3/24)/EI = -\,wL^3/8EI$ (13)

$v'(L_+) \cong (-\,wL^3/8 + 3\,wL^3/8)/EI = wL^3/4EI$ (14)

$v(L) \cong (-\,wL^4/24 + wL^4/12 - 5wL^4/24)/EI = -\,wL^4/6\,EI$ (15)

[Instead of $M(0) = 0$ and $M(L) = 0$, one can use
A is hinged end $\Rightarrow C_2 = 0;\ \sum BM_B = 0 \Rightarrow C_1 \times L - wL \times L/2 = 0 \Rightarrow C_1 = wL/2$]

13. Calculate the reaction at support B [EI = constant].



$EIv^{iv}(x) \cong w(x) = -10\langle x-0\rangle^{-1}{}_{*} + R_B\langle x-5\rangle^{-1}{}_{*}$

$EIv'''(x) \cong V(x) = -10\langle x-0\rangle^{0} + R_B\langle x-5\rangle^{0} + C_1$ (1)

$EIv''(x) \cong M(x) = -10\langle x-0\rangle^{1} + R_B\langle x-5\rangle^{1} + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -5\langle x-0\rangle^{2} + R_B\langle x-5\rangle^{2}/2 + C_1\,x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -5\langle x-0\rangle^{3}/3 + R_B\langle x-5\rangle^{3}/6 + C_1\,x^3/6 + C_2x^2/2 + C_3\,x + C_4$ (4)

$V(0) = 0 \Rightarrow C_1 = 0;\ M(0) = 0 \Rightarrow C_2 = 0$ (5~6)

$D(5) = -\,208.33 + 5\,C_3 + C_4 = 0 \Rightarrow 5C_3 + C_4 = 208.33$ (7)

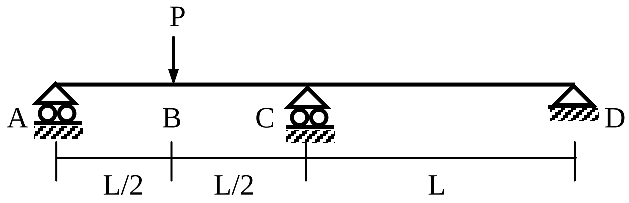
$S(20) = -2000 + 112.5\,R_B + C_3 = 0 \Rightarrow C_3 + 112.5\,R_B = 2000$ (8)

$D(20) = -\,40000/3 + 562.5\,R_B + 20\,C_3 + C_4 = 0 \Rightarrow 20\,C_3 + C_4 + 562.5\,R_B = 13333.33$ (9)

Solving (7~9) $\Rightarrow C_3 = 312.5,\ C_4 = -1354.17,\ R_B = 15$ (10)

[Statically Indeterminate Structure]

14. Calculate the deflection at B [EI = constant].



$EIv^{iv}(x) \cong w(x) = -P<x-L/2>^{-1}_{*} + R_C<x-L>^{-1}_{*}$

$EIv'''(x) \cong V(x) = -P<x-L/2>^0 + R_C<x-L>^0_{*} + C_1$ (1)

$EIv''(x) \cong M(x) = -P<x-L/2>^1 + R_C<x-L>^1 + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -P<x-L/2>^2/2 + R_C<x-L>^2/2 + C_1x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -P<x-L/2>^3/6 + R_C<x-L>^3/6 + C_1x^3/6 + C_2x^2/2 + C_3\,x + C_4$ (4)

$M(0) = 0 \Rightarrow C_2 = 0;\ D(0) = 0 \Rightarrow C_4 = 0$ (5~6)

$M(2L) = -3PL/2 + R_C\,L + 2C_1\,L = 0 \Rightarrow R_C + 2C_1 = 3P/2$ (7)

$D(L) = -\,PL^3/48 + C_1\,L^3/6 + C_3\,L = 0 \Rightarrow C_1\,L^2/6 + C_3 = PL^2/48$ (8)

$D(2L) = -9PL^3/16 + R_C\,L^3/6 + 8C_1\,L^3/6 + 2C_3\,L = 0 \Rightarrow R_C\,L^2/6 + 4C_1\,L^2/3 + 2C_3 = 9PL^2/16$ (9)

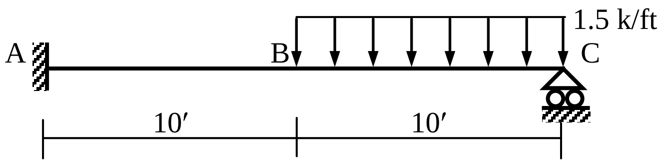
Solving (7~9) $\Rightarrow R_C = 11\ P/16;\ C_1 = 13\ P/32;\ C_3 = -3PL^2/64$ (10)

$EIv(x) = D(x) \cong -P<x-L/2>^3/6 + 11\ P<x-L>^3/96 + 13\ Px^3/192 - 3PL^2\,x/64$ (11)

$v(L/2) \cong (13\ PL^3/1536 - 3\ PL^3/128)/EI = -23\ PL^3/1536\ EI$ (12)

[Statically Indeterminate Structure]

15. Calculate the deflection at B and the rotation at C [EI = 40,000 k-ft^2].



$EIv^{iv}(x) \cong w(x) = -1.5<x-10>^0$

$EIv'''(x) \cong V(x) = -1.5<x-10>^1 + C_1$ (1)

$EIv''(x) \cong M(x) = -1.5<x-10>^2/2 + C_1\,x + C_2$ (2)

$EIv'(x) = S(x) \cong -1.5<x-10>^3/6 + C_1\,x^2/2 + C_2\,x + C_3$ (3)

$EIv(x) = D(x) \cong -1.5<x-10>^4/24 + C_1\,x^3/6 + C_2\,x^2/2 + C_3\,x + C_4$ (4)

$S(0) = 0 \Rightarrow C_3 = 0;\ D(0) = 0 \Rightarrow C_4 = 0$ (5~6)

$M(20) = -75 + 20\ C_1 + C_2 = 0 \Rightarrow 20\ C_1 + C_2 = 75$ (7)

$D(20) = -625 + 1333.33\ C_1 + 200\ C_2 = 0 \Rightarrow 1333.33\ C_1 + 200\ C_2 = 625$ (8)

Solving (7), (8) $\Rightarrow C_1 = 5.39,\ C_2 = -32.81$ (10)

$EIv'(x) = S(x) \cong -1.5<x-10>^3/6 + 5.39\,x^2/2 - 32.81\,x$ (11)

$EIv(x) = D(x) \cong -1.5<x-10>^4/24 + 5.39\,x^3/6 - 32.81\,x^2/2$ (12)

$v'(20) \cong (-250 + 1078 - 656.2)/40000 = 4.297\times10^{-3}$ rad (13)

$v(10) \cong (898.44 - 1640.62)/40000 = -18.55\times10^{-3}$ ft (14)

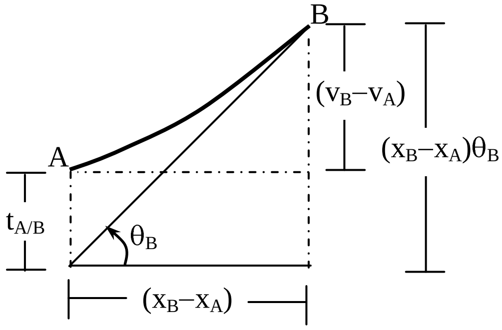
[Statically Indeterminate Structure]

Moment-Area Theorems

The moment-curvature relationship $\Rightarrow$ Curvature = M/EI
For small deflection and slope, Curvature $\cong d^2v/dx^2 = d\theta/dx$
$\therefore d\theta/dx = M/EI \Rightarrow d\theta = (M/EI)\,dx$ ……………………………….(i)
Integrating Eq. (i) between A and B $\Rightarrow \int d\theta = \int (M/EI)\,dx$
$\Rightarrow \theta_B - \theta_A = \int (M/EI)\,dx$, where $\int$ is integration between A and B ………………...….(ii)
Eq. (ii) $\Rightarrow$ The area under (M/EI) diagram between the points A and B is equal to the change of slope between two points. This is the 1st Moment-Area Theorem.

Multiplying both sides of Eq. (i) by x $\Rightarrow x\,d\theta = x\,(M/EI)\,dx$ ……………………(iii)
Integrating Eq. (iii) between A and B $\Rightarrow \int x\,d\theta = \int x\,(M/EI)\,dx$
$\Rightarrow (x_B - x_A)\,\theta_B - (v_B - v_A) = \int x\,(M/EI)\,dx$, where $\int$ is integration between A and B ………...(iv)

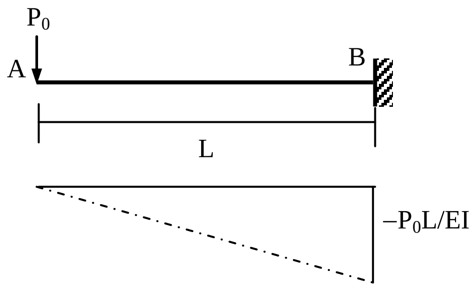
The figure at right shows the geometric significance of various terms in Eq. (iv).
$\therefore$ Eq. (iv) $\Rightarrow$ The moment of the area under (M/EI) diagram between the points A and B (i.e., $\int x\,(M/EI)\,dx$) equals to the deflection of A with respect to the tangent at B; i.e., $t_{A/B}$. This is the 2nd Moment-Area Theorem.



Example 1: Calculate the tip rotation and deflection of the cantilever beam shown below [EI = const].

$\theta_B - \theta_A$ = Area of M/EI diagram between A an B
$= (-P_0L/EI) \times L/2 = -P_0L^2/2EI$
$\Rightarrow \theta_A = P_0L^2/2EI$ …………..(1)

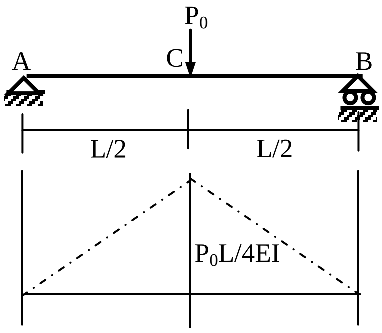
$(x_B - x_A)\,\theta_B - (v_B - v_A) = (-P_0L/EI) \times L/2 \times 2L/3$
$\Rightarrow L \times 0 - 0 + v_A \cong -P_0L^3/3EI$
$\Rightarrow v_A = -P_0L^3/3EI$ …………..(2)



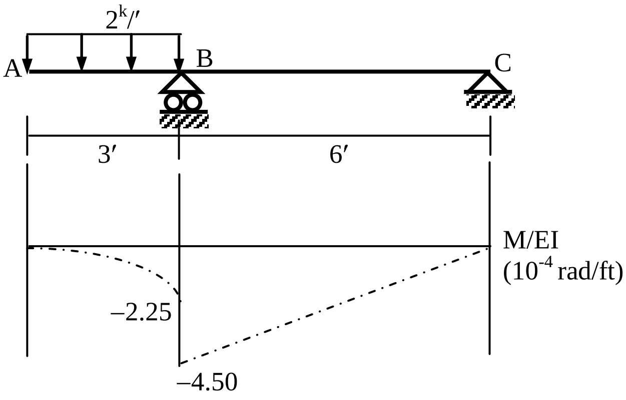
Example 2: Calculate the end rotation and midspan deflection of the simply supported beam shown below.

$\theta_B - \theta_A = (P_0L/4EI) \times L/2 = P_0L^2/8EI$ …………..(1)
$(x_B - x_A)\,\theta_B - (v_B - v_A) = (P_0L/4EI) \times L/2 \times L/2$
$\Rightarrow L\,\theta_B - 0 + 0 = P_0L^3/16EI$
$\Rightarrow \theta_B = P_0L^2/16EI$; $\therefore$ Eq. (1) $\Rightarrow \theta_A = -P_0L^2/16EI$ ………..(2)

$(x_C - x_A)\,\theta_C - (v_C - v_A) = (P_0L/4EI) \times L/4 \times L/3$
$\Rightarrow L/2 \times 0 - v_C + 0 = P_0L^3/48EI$
$\Rightarrow v_C = -P_0L^3/48EI$ ………....(3)



1. Calculate v_A using the moment-area theorems [EI_{AB} = 40,000 k-ft^2, EI_{BC} = 20,000 k-ft^2].

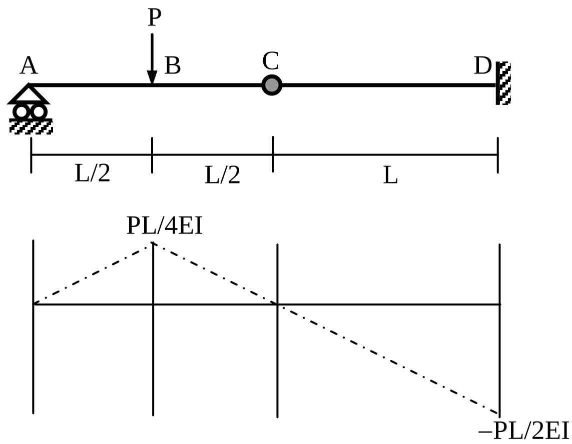


$(x_C - x_B)\,\theta_C - (v_C - v_B) = (-4.50) \times 10^{-4} \times 6/2 \times 6/3$
$\Rightarrow 6\,\theta_C - 0 + 0 = -27 \times 10^{-4}$
$\Rightarrow \theta_C = -4.50 \times 10^{-4}$ rad ....(1)

$\theta_C - \theta_B = (-4.50) \times 10^{-4} \times 6/2$
$\Rightarrow \theta_B = 9.00 \times 10^{-4}$ rad ....(2)

$(x_B - x_A)\,\theta_B - (v_B - v_A) = (-2.25) \times 10^{-4} \times 3/3 \times 2.25$
$\Rightarrow 3 \times 9.00 \times 10^{-4} - 0 + v_A = -5.06 \times 10^{-4}$
$\Rightarrow v_A = -3.21 \times 10^{-3}$ ft = –0.0385 in ....(3)

2. Calculate $\theta_{C(-)}$, $\theta_{C(+)}$ and v_C using the moment-area theorems [EI = constant].

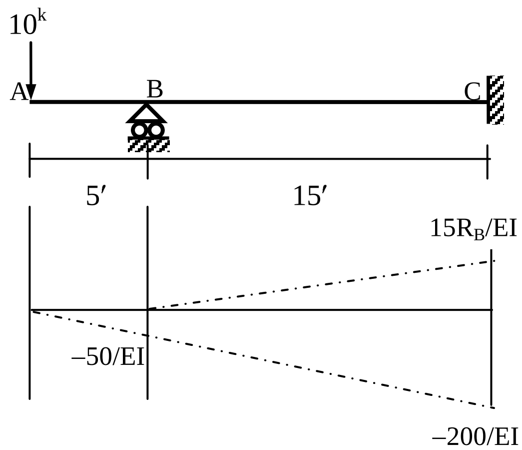


$(x_D - x_C)\,\theta_D - (v_D - v_C) = (-PL/2EI) \times L/2 \times 2L/3$
$\Rightarrow L \times 0 - 0 + v_C = -PL^3/6EI$
$\Rightarrow v_C = -PL^3/6EI$ ....(1)

$\theta_D - \theta_{C(+)} = (-PL/2EI) \times L/2$
$\Rightarrow \theta_{C(+)} = PL^2/4EI$ ....(2)

$(x_C - x_A)\,\theta_{C(-)} - (v_C - v_A) = (PL/4EI) \times L/2 \times L/2$
$\Rightarrow L \times \theta_{C(-)} - (-PL^3/6EI) + 0 = PL^3/16EI$
$\Rightarrow \theta_{C(-)} = -5PL^2/48EI$ ....(3)

3. Calculate R_B and v_A using the moment-area theorems [EI = constant = 40,000 k-ft^2].

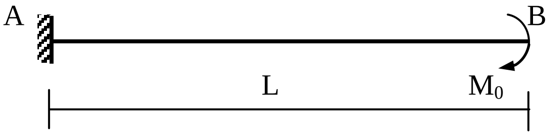


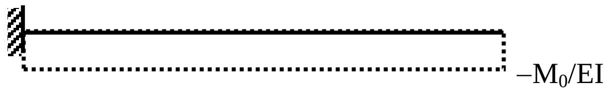
$(x_C - x_B)\,\theta_C - (v_C - v_B)$
$= \{(15\,R_B - 200) \times 15/2 \times 10 - 50 \times 15/2 \times 5\}/EI$
$\Rightarrow 15 \times 0 - 0 + 0 = \{(15\,R_B - 200) - 25\} \times 75/EI$
$\Rightarrow R_B = 15$ kips

$(x_C - x_A)\,\theta_C - (v_C - v_A)$
$= \{225 \times 15/2 \times (10 + 5) - 200 \times 20/2 \times 40/3\}/EI$
$\Rightarrow 20 \times 0 - 0 + v_A = \{(15\,R_B - 200) - 25\} \times 75/EI$
$\Rightarrow v_A = -\,0.0339$ ft $= -\,0.406$ in

Beam Deflection Solutions using Moment-Area Theorems

2. Calculate the deflection and rotation at point B [EI = constant].



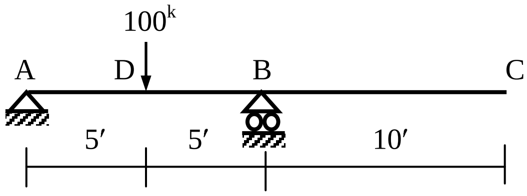


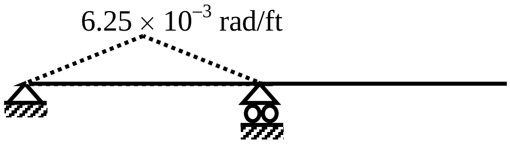
Between A and B, $\theta_A = 0$, $v_A = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_B - \theta_A = L \times (-M_0/EI) = -M_0L/EI \Rightarrow \theta_B = -M_0L/EI$ ……….(1)

and 2nd theorem $\Rightarrow L\theta_B - v_B + v_A = (-M_0\ L/EI) \times L/2 = -M_0L^2/2EI \Rightarrow v_B = -M_0L^2/2EI$ ………(2)

3. Calculate the deflection and rotation at point C [EI = 40,000 k-ft^2].





Between A and B, $v_A = 0$, $v_B = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_B - \theta_A = 10/2 \times 6.25 \times 10^{-3} = 31.25 \times 10^{-3}$ ……….(1)

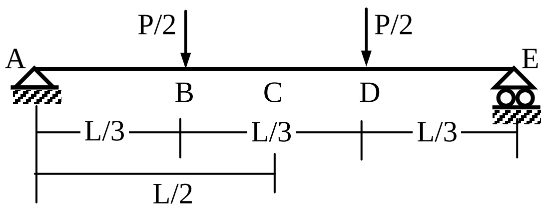
and 2nd theorem $\Rightarrow 10\ \theta_B - v_B + v_A = 31.25 \times 10^{-3} \times 10/2 \Rightarrow \theta_B = 15.63 \times 10^{-3}$ rad ……….(2)

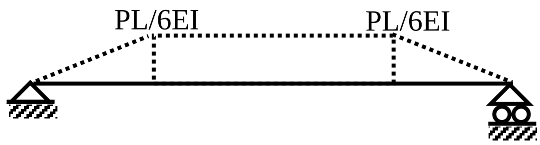
Between B and C, $\theta_B = 15.63 \times 10^{-3}$ rad, $v_B = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_B = 0 \Rightarrow \theta_C = 0.0156$ rad ……….(3)

and 2nd theorem $\Rightarrow 10\ \theta_C - v_C + v_B = 0 \Rightarrow v_C = 0.156$ ft ……….(4)

4. Calculate the deflection at C and the rotation at A [EI = constant].





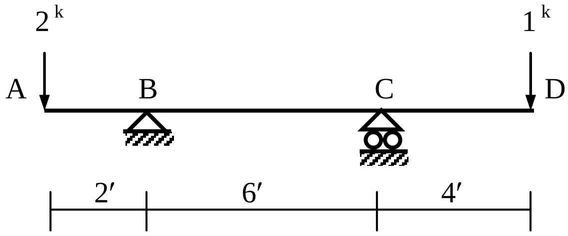
Using symmetry between A and C, $v_A = 0$, $\theta_C = 0$

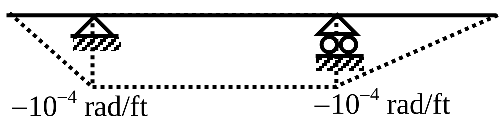
$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_A = L/6 \times PL/6EI + L/6 \times PL/6EI = PL^2/18EI \Rightarrow \theta_A = -PL^2/18EI$ ...…….(1)

and 2nd theorem $\Rightarrow (L/2)\ \theta_C - v_C + v_A = PL^2/36\ EI \times (2L/9) + PL^2/36\ EI \times (L/3 + L/12) = 23\ PL^3/1296\ EI$

$\Rightarrow v_C = -23\ PL^3/1296\ EI$ ……….(2)

5. Calculate the deflection at A and the rotation at B [EI = 40,000 k-ft^2].





Between B and C, $v_B = 0$, $v_C = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_B = 6 \times (-10^{-4}) = -6 \times 10^{-4}$ ……….(1)

and 2nd theorem $\Rightarrow 6\ \theta_C - v_C + v_B = (-6 \times 10^{-4}) \times 6/2 \Rightarrow \theta_C = -3 \times 10^{-4}$ rad ……….(2)

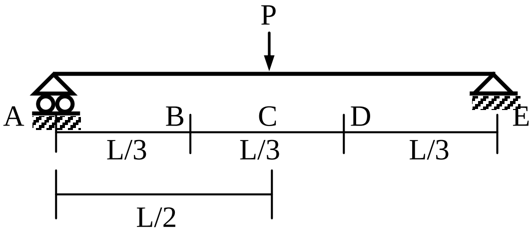
$\therefore (1) \Rightarrow \theta_B = 3 \times 10^{-4}$ rad ...…….(1)′

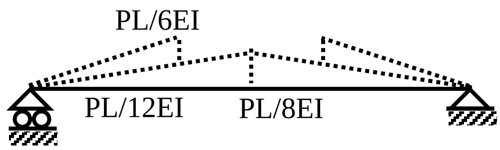
Between A and B, $\theta_B = 3 \times 10^{-4}$ rad, $v_B = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_B - \theta_A = 2/2 \times (-10^{-4}) = -10^{-4} \Rightarrow \theta_A = 4 \times 10^{-4}$ rad ……….(3)

and 2nd theorem $\Rightarrow 2\ \theta_B - v_B + v_A = -10^{-4} \times 4/3 \Rightarrow v_A = -7.33 \times 10^{-4}$ ft ……….(4)

7. Calculate the deflection at C [$EI_{AB} = EI_{DE} = EI$, $EI_{BCD} = 2EI$].





Using symmetry between A and C, $v_A = 0$, $\theta_C = 0$

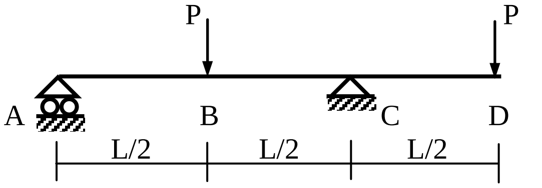
$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_A = L/4 \times PL/8EI + L/6 \times PL/12EI = PL^2/32EI + PL^2/72EI$

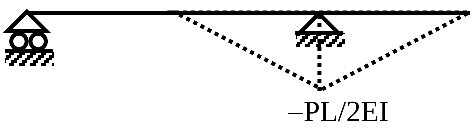
$\Rightarrow \theta_A = -13PL^2/288EI$ ……….(1)

and 2nd theorem $\Rightarrow (L/2)\,\theta_C - v_C + v_A = PL^2/32\,EI \times (L/3) + PL^2/72\,EI \times (2L/9) = 35\,PL^3/2592\,EI$

$\Rightarrow v_C = -35PL^3/2592\,EI$ ……….(2)

8. Calculate the deflection at D [EI = constant].





Between A and C, $v_A = 0$, $v_C = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_A = L/4 \times (-PL/2EI) = -PL^2/8EI$ ……….(1)

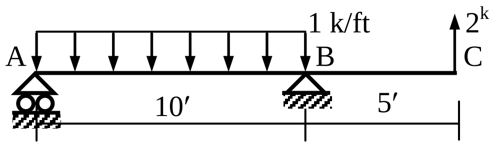
and 2nd theorem $\Rightarrow L\theta_C - v_C + v_A = (-PL^2/8EI) \times (L/2 + L/3) \Rightarrow \theta_C = -5PL^2/48EI$ ……….(2)

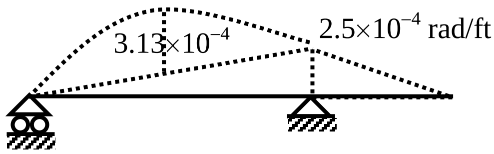
Between C and D, $\theta_C = -5PL^2/48EI$, $v_C = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_D - \theta_C = L/4 \times (-PL/2EI) = -PL^2/8EI \Rightarrow \theta_D = -11\,PL^2/48\,EI$ ……….(3)

and 2nd theorem $\Rightarrow (L/2)\,\theta_D - v_D + v_C = (-PL^2/8EI) \times L/6 \Rightarrow v_D = -3PL^3/32\,EI$ ……….(4)

9. Calculate the deflection at C and the rotation at B [EI = 40,000 k-ft^2].





Between A and B, $v_A = 0$, $v_B = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_B - \theta_A = 10/2 \times 2.5\times10^{-4} + 20/3 \times 3.13 \times 10^{-4} = (1.25 + 2.08)\times10^{-3} = 3.33 \times 10^{-3}$…....(1)

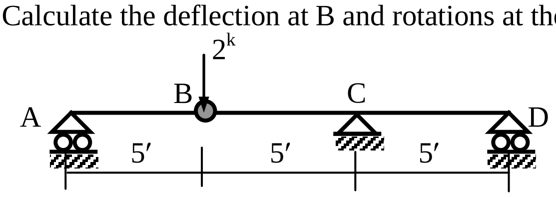
and 2nd theorem $\Rightarrow 10\theta_B - v_B + v_A = 1.25 \times 10^{-3} \times 20/3 + 2.08 \times 10^{-3} \times 5 \Rightarrow \theta_B = 1.875\times10^{-3}$ rad ...…(2)

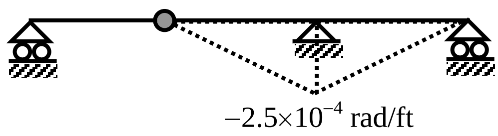
Between B and C, $\theta_B = 1.875\times10^{-3}$ rad, $v_B = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_B = 5/2 \times 2.5\times10^{-4} = 6.25\times10^{-4} \Rightarrow \theta_C = 2.5 \times 10^{-3}$ rad ……….(3)

and 2nd theorem $\Rightarrow 5\,\theta_C - v_C + v_B = 6.25\times10^{-4} \times 5/3 \Rightarrow v_C = 11.46 \times 10^{-3}$ ft ……….(4)

10. Calculate the deflection at B and rotations at the left and right of B [EI = 40,000 k-ft^2].





Between C and D, $v_C = 0$, $v_D = 0$

$\therefore$ 1st theorem $\Rightarrow \theta_D - \theta_C = 5/2 \times (-2.5\times10^{-4}) = -6.25 \times 10^{-4}$ ………....(1)

and 2nd theorem $\Rightarrow 5\theta_D - v_D + v_C = -6.25 \times 10^{-4} \times 5/3 \Rightarrow \theta_D = -2.08 \times 10^{-4}$ rad ……..….(2)

$\therefore (1) \Rightarrow \theta_C = 4.17 \times 10^{-4}$ rad ...…….(1)′

Between B and C, $\theta_C = 4.17\times10^{-4}$ rad, $v_C = 0$

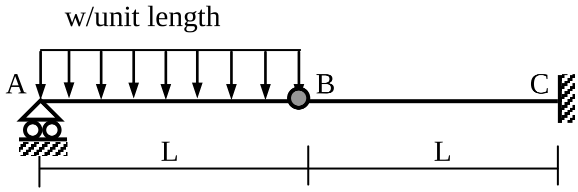
$\therefore$ 1st theorem $\Rightarrow \theta_C - \theta_{B(+)} = 5/2 \times (-2.5\times10^{-4}) = -6.25\times10^{-4} \Rightarrow \theta_{B(+)} = 1.04 \times 10^{-3}$ rad ……….(3)

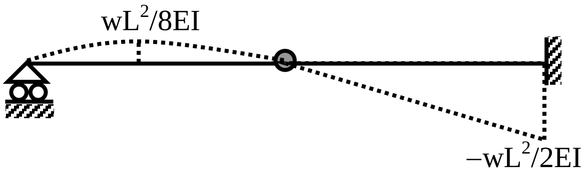
and 2nd theorem $\Rightarrow 5\theta_C - v_C + v_B = -6.25\times10^{-4} \times 10/3 \Rightarrow v_B = -4.17 \times 10^{-3}$ ft ……….(4)

Between A and B, $v_A = 0$, $v_B = -4.17\times10^{-3}$ ft

$\therefore$ 2nd theorem $\Rightarrow 5\theta_{B(-)} - v_B + v_A = 0 \Rightarrow \theta_{B(-)} = -8.33 \times 10^{-4}$ rad ……….(5)

12. Calculate the deflection at B and rotations at the left and right of B [EI = constant].





Between B and C, $\theta_C = 0$, $v_C = 0$

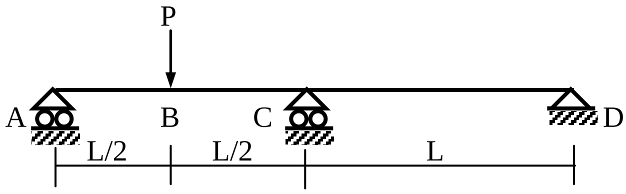
$\therefore$ 1st theorem$\Rightarrow$ $\theta_C - \theta_{B(+)} = L/2 \times(-wL^2/2EI) = -wL^3/4EI \Rightarrow \theta_{B(+)} = wL^3/4EI$ ....(1)

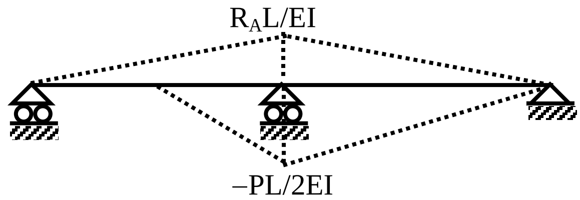
and 2nd theorem $\Rightarrow L\theta_C - v_C + v_B = -wL^3/4EI \times 2L/3 \Rightarrow v_B = -wL^4/6EI$ ....(2)

Between A and B, $v_A = 0$, $v_B = -wL^4/6EI$

$\therefore$ 2nd theorem $\Rightarrow L\theta_{B(-)} - v_B + v_A = 2L/3\times (wL^2/8EI)\times L/2 \Rightarrow \theta_{B(-)} = -wL^3/8EI$ ....(3)

14. Calculate the deflection at B [EI = constant].





Between A and C, $v_A = 0$, $v_C = 0$

$\therefore$ 1st theorem$\Rightarrow$ $\theta_C - \theta_A = L/2 \times(R_AL/EI) + L/4 \times (-PL/2EI) = R_AL^2/2EI - PL^2/8EI$ ....(1)

and 2nd theorem $\Rightarrow L\theta_C - v_C + v_A = R_AL^2/2EI\times 2L/3 - PL^2/8EI\times 5L/6 \Rightarrow \theta_C = R_AL^2/3EI - 5PL^2/48EI$......(2)

Between C and D, $v_C = 0$, $v_D = 0$

$\therefore$ 1st theorem$\Rightarrow$ $\theta_D - \theta_C = L/2 \times(R_AL/EI) + L/2 \times (-PL/2EI) = R_AL^2/2EI - PL^2/4EI$ ....(3)

and 2nd theorem $\Rightarrow L\theta_D - v_D + v_C = R_AL^2/2EI\times L/3 - PL^2/4EI\times L/3 \Rightarrow \theta_D = R_AL^2/6EI - PL^2/12EI$ ....(4)

$\therefore$ (2~4) $\Rightarrow (R_AL^2/6EI - PL^2/12EI) - (R_AL^2/3EI - 5PL^2/48EI) = R_AL^2/2EI - PL^2/4EI$

$\Rightarrow -2R_AL^2/3EI = -13PL^2/48EI \Rightarrow R_A = 13\ P/32$ ....(5)

$\therefore$ (2) $\Rightarrow \theta_C = PL^2/32EI$ ....(6)

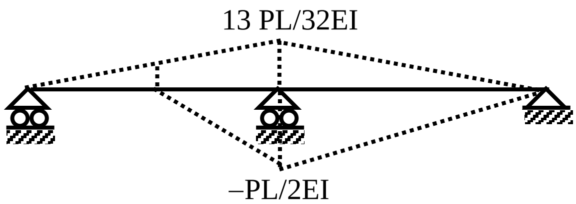
Between B and C, $\theta_C = PL^2/32EI$, $v_C = 0$

$\therefore$ 2nd theorem $\Rightarrow (L/2)\theta_C - v_C + v_B = -PL^2/8EI\times L/3$

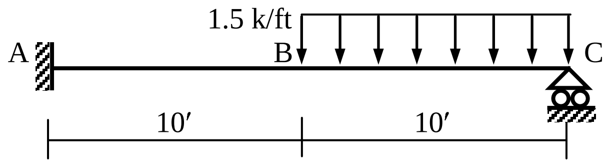
$+13PL^2/128EI\times L/4 + 13PL^2/256EI\times L/3$

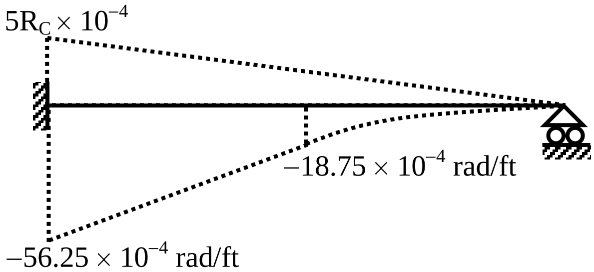
$= PL^3/1536\ EI$

$\Rightarrow v_B = -23\ PL^3/1536\ EI$ ....(7)



15. Calculate the deflection at B and the rotation at C [EI = 40,000 k-ft^2].





Between A and C, $\theta_A = 0$, $v_A = 0$, $v_C = 0$

$\therefore$ 1st theorem$\Rightarrow$ $\theta_C - \theta_A = 10 \times (-18.75 \times 10^{-4}) + 10/2 \times (-37.5 \times 10^{-4}) + 10/3 \times (-18.75 \times 10^{-4})$

$+ 20/2 \times (5R_C \times 10^{-4}) = -187.5 \times 10^{-4} - 187.5 \times 10^{-4} - 62.5 \times 10^{-4} + 50R_C \times 10^{-4}$

$\Rightarrow \theta_C = -437.5 \times 10^{-4} + 50\ R_C \times 10^{-4}$ ....(1)

and 2nd theorem$\Rightarrow$ $20\ \theta_C - v_C + v_A = -187.5 \times 10^{-4}\times 5 - 187.5 \times 10^{-4}\times 10/3 - 62.5 \times 10^{-4}\times 12.5$

$+ 50R_C \times 10^{-4}\times 20/3$

$\Rightarrow \theta_C = -46.88\times10^{-4} - 31.25\times10^{-4} - 39.06\times10^{-4} + 16.67R_C\times10^{-4} = -117.19\times10^{-4} + 16.67R_C\times10^{-4}$ ......(2)

$\therefore$ (1, 2) $\Rightarrow -437.5 \times 10^{-4} + 50\ R_C \times 10^{-4} = -117.19 \times 10^{-4} + 16.67\ R_C \times 10^{-4} \Rightarrow R_C = 9.61$ kips ......(3)

$\therefore$ (1, 3) $\Rightarrow \theta_C = -437.5 \times 10^{-4} + 480.5 \times 10^{-4} = 4.297 \times 10^{-3}$ rad ....(4)

Between B and C, $\theta_C = 4.297 \times 10^{-3}$ rad, $v_C = 0$

$\therefore$ 2nd theorem $\Rightarrow 10\ \theta_C - v_C + v_B = 10/2\times(24.03 \times 10^{-4})\times 10/3 + 10/3\times(-18.75 \times 10^{-4})\times 10/4 = 244.25 \times 10^{-4}$

$\Rightarrow v_B = -18.55 \times 10^{-3}$ ft ....(5)

Conjugate Beam Method

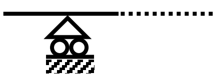
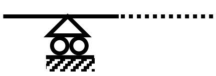
From moment-curvature relationship, Curvature $\kappa = M/EI \cong d^2v/dx^2$ …………… (i)

$\therefore$ Slope $\theta = dv/dx \cong \int (M/EI)\, dx$, and Deflection $v = \int \theta\, dx \cong \int\{\int (M/EI)\, dx\}\, dx$ ………….. (ii)

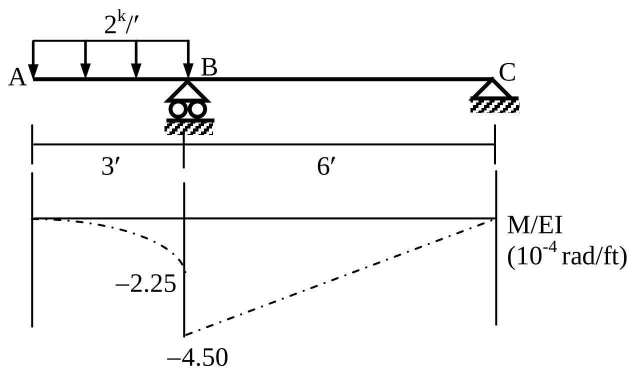
On the other hand, if w is the load per unit length,

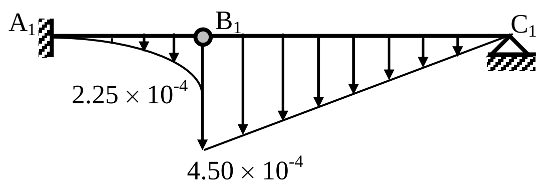
Shear Force $V = \int w\, dx$ and Bending Moment $M = \int V\, dx = \int\{\int w\, dx\}\, dx$ ………….. (iii)

From the analogy of equations (ii) and (iii), if w is replaced by M/EI, the shear force V and bending moment M can be considered to be equivalent to slope θ and deflection v respectively.
However, such equivalence should be represented in the support conditions as well, which should be modified to form a 'conjugate' of the original beam.

	Original Beam	Conjugate Beam
(a)		
	Free end: $\theta \neq 0$, $v \neq 0$	Fixed end: $V \neq 0$, $M \neq 0$
(b)		
	Fixed end: $\theta = 0$, $v = 0$	Free end: $V = 0$, $M = 0$
(c)		
	Hinged/Roller end: $\theta \neq 0$, $v = 0$	Hinged/Roller end: $V \neq 0$, $M = 0$
(d)		
	Internal Support: $\theta \neq 0$, $v = 0$	Internal Hinge: $V \neq 0$, $M = 0$
(e)		
	Internal Hinge: $\theta \neq 0$, $v \neq 0$	Internal Support: $V \neq 0$, $M \neq 0$

1. Calculate θ_C, θ_A, v_A using Conjugate Beam Method [$EI_{AB} = 40\times10^3$ k-ft^2, $EI_{BC} = 20\times10^3$ k-ft^2].





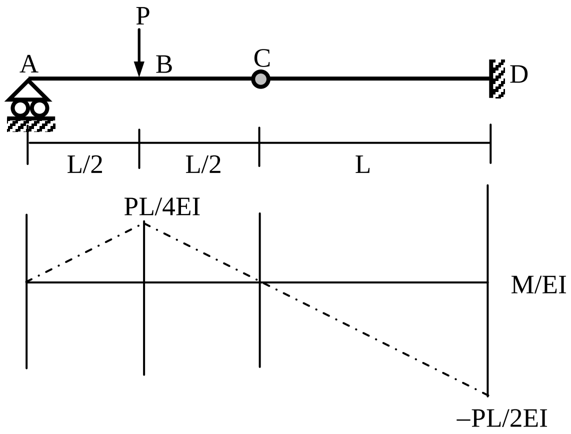
Conjugate Beam loaded by M/EI

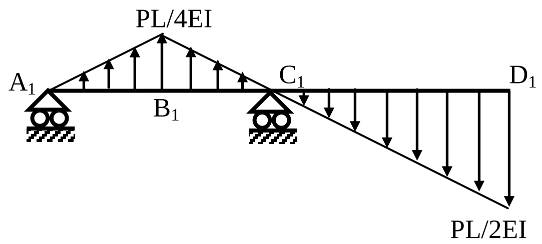
$BM_{B1} = 0$ (right) $\Rightarrow -R_{C1} \times 6 + 4.50 \times 10^{-4} \times 6/2 \times 6/3 = 0 \Rightarrow R_{C1} = 4.50\times 10^{-4}$ rad .... ….(1)
$\sum F_y = 0 \Rightarrow R_{A1} - 2.25 \times 10^{-4} \times 3/3 - 4.50 \times 10^{-4} \times 6/2 + R_{C1} = 0 \Rightarrow R_{A1} = 11.25 \times 10^{-4}$ rad ….(2)

$BM_{B1} = 0$ (left) $\Rightarrow M_{A1} + R_{A1} \times 3 - 2.25 \times 10^{-4} \times 3/3 \times (1/4 \times 3) = 0$
$\Rightarrow M_{A1} = -32.06 \times 10^{-4}$ ft $= -0.0385$ in ……….....……(3)

$\therefore \theta_C = V_{C1} = -R_{C1} = 4.50\times 10^{-4}$ rad, $\theta_A = V_{A1} = R_{A1} = 11.25\times 10^{-4}$ rad
$v_A = BM_{A1} = M_{A1} = -0.0385$ in

2. Calculate $\theta_{C(-)}$, $\theta_{C(+)}$ and v_C using the Conjugate Beam Method [EI = constant].





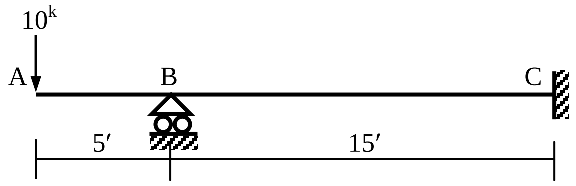
Conjugate Beam loaded by M/EI

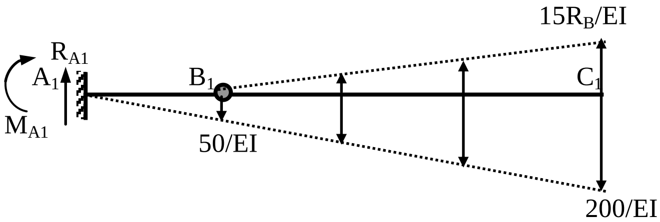
$\sum F_{y(C1)} = 0 \Rightarrow V_{C1(+)} - (PL/2EI) \times L/2 = 0 \Rightarrow V_{C1(+)} = PL^2/4EI$ ………..(1)
$\sum M_{C1} = 0$ (right) $\Rightarrow M_{C1} + (PL/2EI) \times L/2 \times 2L/3 = 0 \Rightarrow M_{C1} = -PL^3/6EI$ ....… ….(2)

$\sum M_{A1} = 0$ (left of C_1)
$\Rightarrow -M_{C1} - (PL/4EI) \times L/2 \times L/2 + V_{C1(-)} \times L = 0 \Rightarrow V_{C1(-)} = -5PL^2/48EI$ ....… ….(3)

$\therefore \theta_{C(+)} = V_{C1(+)} = PL^2/4EI$, $\theta_{C(-)} = V_{C1(-)} = -5PL^2/48EI$
$v_C = M_{C1} = -PL^3/6EI$

3. Calculate the reaction at support B [EI = constant].

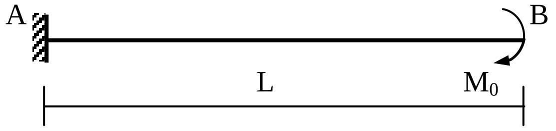


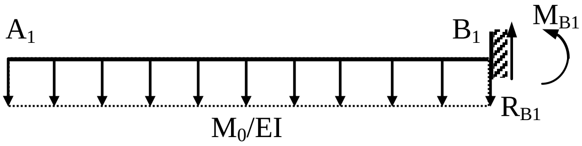


$\therefore$ In the conjugate beam, $BM_{B1} = 0 \Rightarrow -15/2 \times 15\ R_B/EI\times10 + 15\times50/EI \times15/2+15/2 \times150/EI \times 10 = 0$
$\Rightarrow R_B = 15$ kips ……….(1)

Beam Deflection Solutions using Conjugate Beam Method

2. Calculate the deflection and rotation at point B [EI = constant].



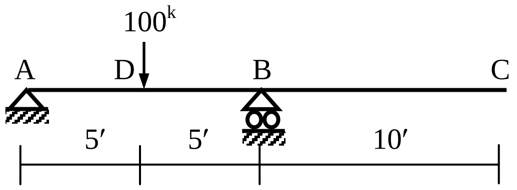


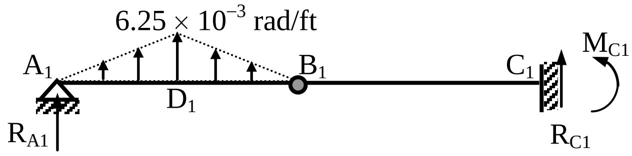
$\therefore$ In the conjugate beam, $\sum F_y = 0 \Rightarrow R_{B1} - L \times (M_0/EI) = 0 \Rightarrow R_{B1} = M_0L/EI$ ……….(1)

and $\sum M_{B1} = 0 \Rightarrow M_{B1} + M_0L/EI \times L/2 = 0 \Rightarrow M_{B1} = -M_0L^2/2EI$ ………(2)

$\therefore \theta_B = V_{B1} = -R_{B1} = -M_0L/EI$, and $v_B = M_{B1} = -M_0L^2/2EI$

3. Calculate the deflection and rotation at point C [EI = 40,000 k-ft^2].





In the conjugate beam, $BM_{B1} = 0 \Rightarrow R_{A1} \times 10 + 10/2 \times 6.25 \times 10^{-3} \times 10/2 = 0$

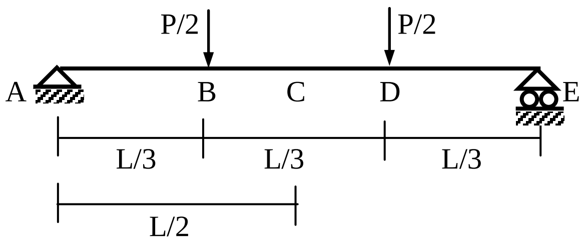
$\Rightarrow R_{A1} = -15.63 \times 10^{-3}$ rad ……....(1)

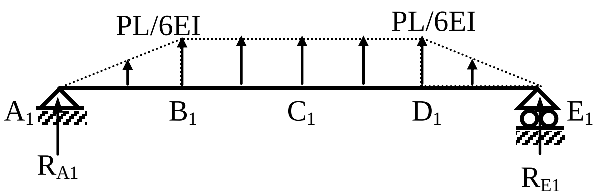
$\therefore \sum F_y = 0 \Rightarrow R_{A1} + 10/2 \times 6.25 \times 10^{-3} + R_{C1} = 0 \Rightarrow R_{C1} = -15.63 \times 10^{-3}$ rad ……….(2)

$\therefore \sum M_{C1} = 0 \Rightarrow R_{A1} \times 20 + 10/2 \times 6.25 \times 10^{-3} \times 15 + M_{C1} = 0 \Rightarrow M_{C1} = 0.156$ ft ……….(3)

$\therefore \theta_C = V_{C1} = -R_{C1} = 15.63 \times 10^{-3}$ rad, and $v_C = M_{C1} = 0.156$ ft

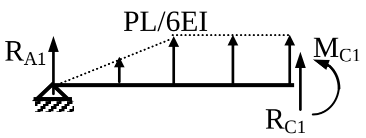
4. Calculate the deflection at C and the rotation at A [EI = constant].





In the conjugate beam, $\sum M_{E1} = 0 \Rightarrow R_{A1} \times L + \{(2L/3)/2 \times PL/6EI + L/3 \times PL/6EI\} \times L/2$

$\Rightarrow R_{A1} = -PL^2/18\ EI$ ………..(1)



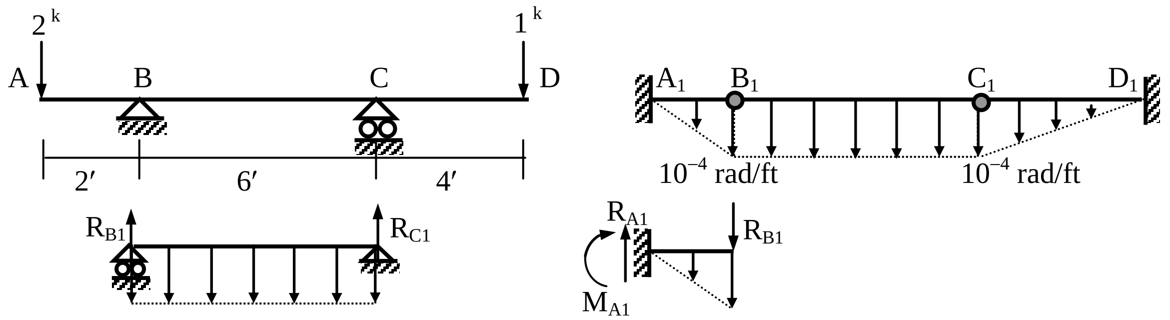
$\therefore$ Section at $C_1 \Rightarrow R_{A1} \times L/2 + \{(L/3)/2 \times PL/6EI \times (L/6 + L/9) + L/6 \times PL/6EI \times L/12\} - M_{C1} = 0$

$\Rightarrow -PL^3/36\ EI + 5PL^3/648\ EI + PL^3/432\ EI - M_{C1} = 0$

$\Rightarrow M_{C1} = -23PL^3/1296\ EI$ ………...(2)

$\therefore \theta_A = V_{A1} = R_{A1} = -PL^2/18EI$, and $v_C = M_{C1} = -23\ PL^3/1296\ EI$

5. Calculate the deflection at A and the rotation at B [EI = 40,000 k-ft^2].



In the portion B_1C_1 of the conjugate beam, $R_{B1} = R_{c1} = 6/2 \times 10^{-4} = 3 \times 10^{-4}$ rad ……..…..(1)

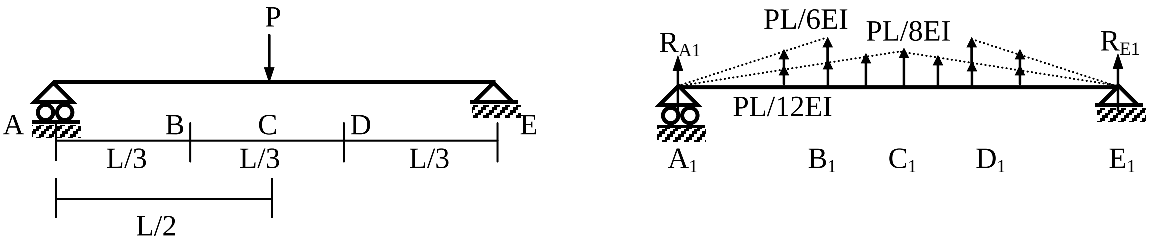
$\therefore$ In the portion A_1B_1, $\sum F_y = 0 \Rightarrow R_{A1} - 2/2 \times 10^{-4} - R_{B1} = 0 \Rightarrow R_{A1} = 4 \times 10^{-4}$ rad ……….(2)

and $\sum M_{A1} = 0 \Rightarrow R_{B1} \times 2 + 2/2 \times 10^{-4} \times 4/3 + M_{A1} = 0 \Rightarrow M_{A1} = -7.33 \times 10^{-4}$ ft ……….(3)

$\therefore \theta_B = V_{B1} = R_{B1} = 3 \times 10^{-4}$ rad, and $v_A = M_{A1} = -7.33 \times 10^{-4}$ ft

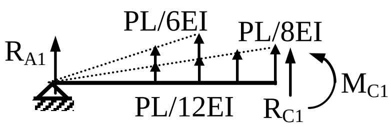
6. Solution in Class Note.

7. Calculate the deflection at C [$EI_{AB} = EI_{DE} = EI$, $EI_{BCD} = 2EI$].



In the conjugate beam, $\sum M_{E1} = 0 \Rightarrow R_{A1} \times L + \{(2L/3)/2 \times PL/12EI + L/2 \times PL/8EI\} \times L/2$

$\Rightarrow R_{A1} = -13\ PL^2/288\ EI$ ………..(1)



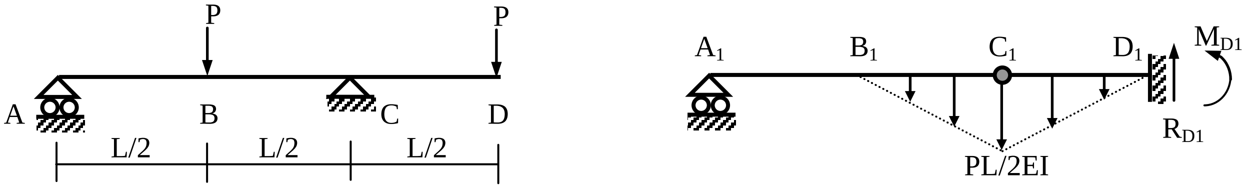
$\therefore$ Section at $C_1 \Rightarrow R_{A1} \times L/2 + \{(L/3)/2 \times PL/12EI \times (L/6 + L/9) + (L/2)/2 \times PL/8EI \times L/6\} - M_{C1} = 0$

$\Rightarrow -13\ PL^3/576\ EI + 5PL^3/1296\ EI + PL^3/192\ EI - M_{C1} = 0$

$\Rightarrow M_{C1} = -35\ PL^3/2592\ EI$ ………...(2)

$\therefore \theta_A = V_{A1} = R_{A1} = -13\ PL^2/288EI$, and $v_C = M_{C1} = -35\ PL^3/2592\ EI$

8. Calculate the deflection at D [EI = constant].

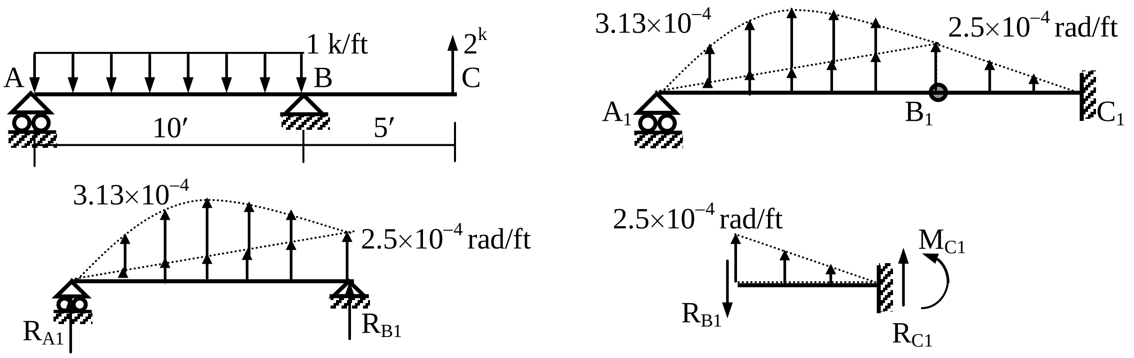


In the conjugate beam, $BM_{C1} = 0 \Rightarrow R_{A1} \times L - (L/2)/2 \times (PL/2EI) \times L/6 = 0 \Rightarrow R_{A1} = PL^2/48\ EI$ …….(1)

$\therefore \sum M_{D1} = 0 \Rightarrow R_{A1} \times 3L/2 - L/2 \times (PL/2EI) \times L/2 - M_{D1} = 0 \Rightarrow M_{D1} = -3PL^3/32\ EI$ ……….(2)

$\therefore v_D = M_{D1} = -3PL^3/32\ EI$

9. Calculate the deflection at C and the rotation at B [EI = 40,000 k-ft^2].



$\therefore$ In the portion A_1B_1, $\sum M_{A1} = 0 \Rightarrow -R_{B1} \times 10 - 10/2 \times 2.5\times10^{-4} \times 20/3 - 20/3 \times 3.13\times10^{-4} \times 5 = 0$

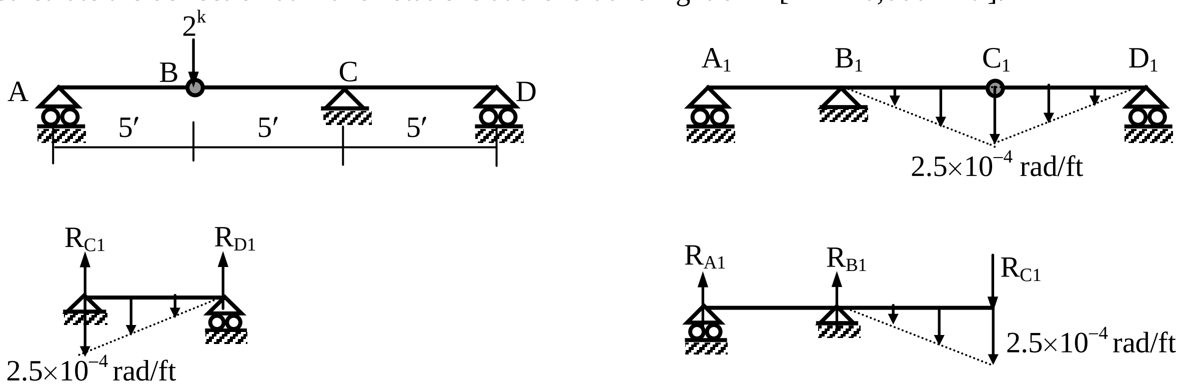
$\Rightarrow R_{B1} = -1.875 \times 10^{-3}$ rad ……….(1)

$\therefore$ In the portion B_1C_1, $\sum M_{C1} = 0 \Rightarrow -R_{B1} \times 5 + 5/2 \times 2.5\times10^{-4} \times 10/3 - M_{C1} = 0$

$\Rightarrow M_{C1} = 11.46 \times 10^{-3}$ ft ……….(2)

$\therefore \theta_B = V_{B1} = -R_{B1} = 1.875 \times 10^{-3}$ rad, and $v_C = M_{C1} = 11.46 \times 10^{-3}$ ft

10. Calculate the deflection at B and rotations at the left and right of B [EI = 40,000 k-ft^2].



$\therefore$ In the portion C_1D_1, $\sum M_{D1} = 0 \Rightarrow R_{C1} \times 5 - 5/2 \times 2.5\times10^{-4} \times 10/3 = 0 \Rightarrow R_{C1} = 4.17 \times 10^{-4}$ rad …..(1)

$\therefore$ In the portion $A_1B_1C_1$, $\sum M_{A1} = 0 \Rightarrow -R_{B1} \times 5 + 5/2 \times 2.5\times10^{-4} \times (5 + 10/3) + R_{C1} \times 10 = 0$

$\Rightarrow R_{B1} = 1.875 \times 10^{-3}$ rad ……….(2)

For a section at right of B_1, $\sum F_y = 0 \Rightarrow V_{B1(+)} - 5/2 \times 2.5\times10^{-4} - R_{C1} = 0 \Rightarrow V_{B1(+)} = 1.04\times10^{-3}$ rad …(3)

and $\sum M_{B1} = 0 \Rightarrow M_{B1} + 5/2 \times 2.5\times10^{-4} \times 10/3 + R_{C1} \times 5 = 0$

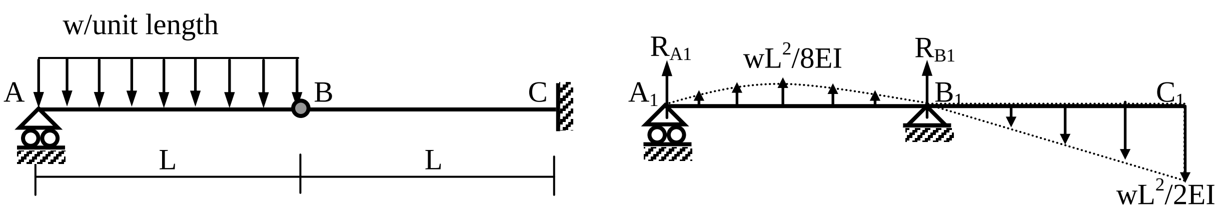
$\Rightarrow M_{B1} = -4.17\times10^{-3}$ ft ……….(4)

For a section at left of B_1, $\sum F_y = 0 \Rightarrow V_{B1(-)} + R_{B1} = V_{B1(+)} \Rightarrow V_{B1(-)} = -8.33 \times 10^{-4}$ rad ……….(5)

$\therefore \theta_{B(-)} = V_{B1(-)} = -8.33 \times 10^{-4}$ rad, $\theta_{B(+)} = V_{B1(+)} = 1.04 \times 10^{-4}$ rad, and $v_B = M_{B1} = -4.17\times10^{-3}$ ft

11. Solution in Class Note.

12. Calculate the deflection at B and rotations at the left and right of B [EI = constant].



$\therefore$ In the conjugate beam, $\sum M_{A1} = 0 \Rightarrow -R_{B1} \times L - 2L/3 \times wL^2/8EI \times L/2 + L/2 \times wL^2/2EI \times (L+2L/3) = 0$

$\Rightarrow R_{B1} = 3wL^3/8EI$ ……….(1)

For a section at right of B_1, $\sum F_y = 0 \Rightarrow V_{B1(+)} - L/2 \times wL^2/2EI = 0 \Rightarrow V_{B1(+)} = wL^3/4EI$ ……….(2)

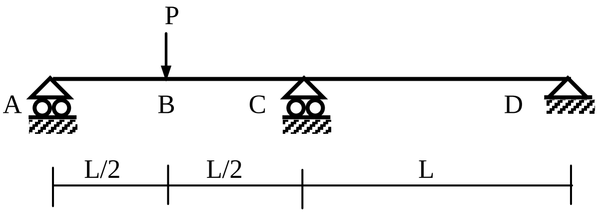
and $\sum M_{B1} = 0 \Rightarrow M_{B1} + L/2 \times wL^2/2EI \times 2L/3 = 0 \Rightarrow M_{B1} = -wL^4/6EI$ ……..(3)

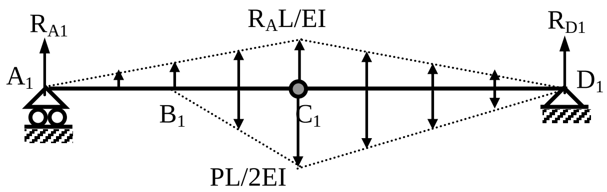
For a section at left of B_1, $\sum F_y = 0 \Rightarrow V_{B1(-)} + R_{B1} = V_{B1(+)} \Rightarrow V_{B1(-)} = -wL^3/8EI$ ……….(4)

$\therefore \theta_{B(-)} = V_{B1(-)} = -wL^3/8EI$, $\theta_{B(+)} = V_{B1(+)} = wL^3/4EI$, and $v_B = M_{B1} = -wL^4/6EI$

13. Solution in Class Note.

14. Calculate the deflection at B [EI = constant].





$\therefore$ In the conjugate beam, $BM_{C1} = 0 \Rightarrow R_{A1} \times L + L/2 \times R_AL/EI \times L/3 - L/4 \times PL/2EI \times L/6 = 0$

$\Rightarrow R_{A1} = -R_AL^2/6EI + PL^2/48EI$ ……….(1)

Also, $BM_{C1} = 0 \Rightarrow -R_{D1} \times L - L/2 \times R_AL/EI \times L/3 + L/2 \times PL/2EI \times L/3 = 0$

$\Rightarrow R_{D1} = -R_AL^2/6EI + PL^2/12EI$ ……….(2)

$\therefore \sum F_y = 0 \Rightarrow R_{A1} + R_{D1} = -(2L/2) \times R_AL/EI + 3L/4 \times PL/2EI = -R_AL^2/EI + 3\,PL^2/8EI$ ………..(3)

$\therefore$ [(1) + (2)], (3) $\Rightarrow -R_AL^2/3EI + 5PL^2/48EI = -R_AL^2/EI + 3PL^2/8EI$

$\Rightarrow 2R_AL^2/3EI = 13\,PL^2/48EI \Rightarrow R_A = 13\,P/32$ ……….(4)

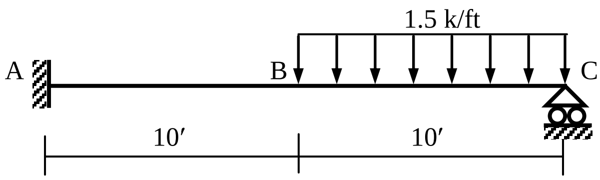
$\therefore$ (1) $\Rightarrow R_{A1} = -R_AL^2/6EI + PL^2/48EI = -13PL^2/192EI + PL^2/48EI = -3PL^2/64EI$ ……….(5)

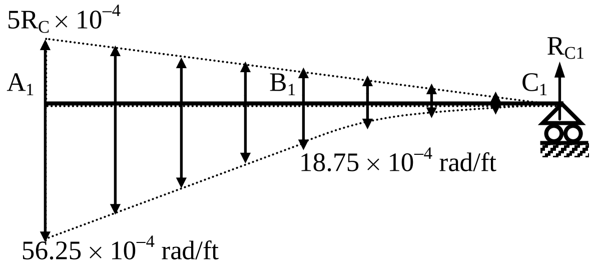
$\therefore$ Section at $B_1 \Rightarrow R_{A1} \times L/2 + L/4 \times R_AL/2EI \times L/6 - M_{B1} = 0$

$\Rightarrow M_{B1} = -3PL^3/128EI + 13PL^3/1536EI = -23PL^3/1536EI$ ……….(6)

$\therefore v_B = M_{B1} = -23PL^3/1536EI$

15. Calculate the deflection at B and the rotation at C [EI = 40,000 k-ft^2].





$\therefore$ In the conjugate beam, $\sum M_{C1} = 0 \Rightarrow -20/2 \times 5R_C \times 10^{-4} \times 40/3 + 10/3 \times 18.75 \times 10^{-4} \times 30/4$

$+ 10 \times 18.75 \times 10^{-4} \times (10+5) + 10/2 \times 37.5 \times 10^{-4} \times (10 + 20/3) = 0$

$\Rightarrow 666.67\,R_C = 6406.25 \Rightarrow R_C = 9.61$ kips ……….(1)

$\therefore \sum F_y = 0 \Rightarrow R_{C1} + 20/2 \times 5 \times 9.61 \times 10^{-4} - 10/3 \times 18.75 \times 10^{-4} - 10/2 \times (18.75 + 56.25) \times 10^{-4} = 0$

$\Rightarrow R_{C1} = -4.297 \times 10^{-3}$ rad ………..(2)

$\therefore$ Section at B_1 (right)

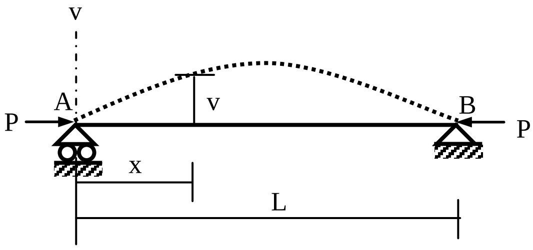
$\Rightarrow -R_{C1} \times 10 - 10/2 \times 2.5 \times 9.61 \times 10^{-4} \times 10/3 + 10/3 \times 18.75 \times 10^{-4} \times 10/4 + M_{B1} = 0$

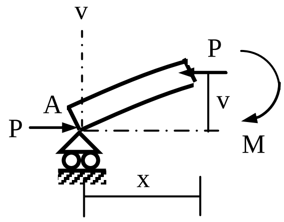
$\Rightarrow M_{B1} = -4.297 \times 10^{-3} \times 10 + 10/2 \times 2.5 \times 9.61 \times 10^{-4} \times 10/3 - 10/3 \times 18.75 \times 10^{-4} \times 10/4$ …….(3)

$= -18.55 \times 10^{-3}$ ft

$\therefore \theta_C = V_{C1} = -R_{C1} = 4.297 \times 10^{-3}$ rad, $v_B = M_{B1} = -18.55 \times 10^{-3}$ ft

Buckling of Columns





$EI\,v'' = -Pv \Rightarrow EI\,v'' + Pv = 0$

$\Rightarrow v'' + \lambda^2 v = 0$ ……………...(i) [where $\lambda^2 = P/EI$]

$\Rightarrow v(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$ ……………...(ii)

$\therefore v(0) = 0 \Rightarrow C_1 = 0$ ...…………(iii)

$v(L) = 0 \Rightarrow C_2 \sin(\lambda L) = 0$

$\therefore$ Either $C_2 = 0 \Rightarrow v(x) = C_1 \cos \lambda x + C_2 \sin \lambda x = 0$ …………...(iv), which is the trivial solution

Or $\lambda L = n\pi \Rightarrow \lambda = n\pi/L$ ...…………(v)

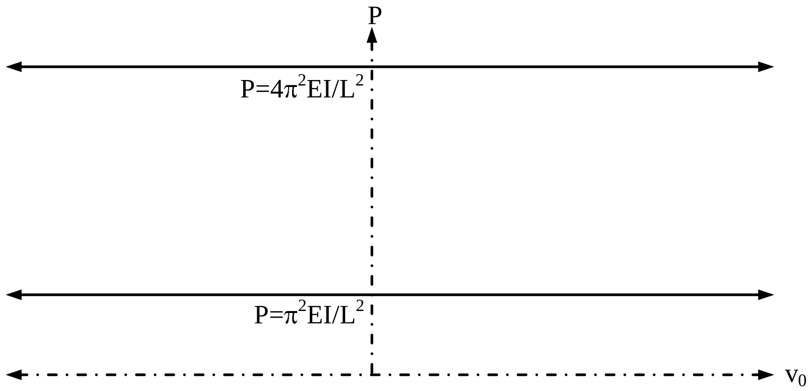
$\therefore P = \lambda^2 EI = n^2\pi^2 EI/L^2$ ...…………(vi)

Eq. (vi) provides a set of solutions for the load P in order to cause deflection of the column. The smallest of these forces is obtained by putting n = 1, resulting in the critical load of the column as

$\therefore P_{cr} = \pi^2 EI/L^2$ ...…………(vii)

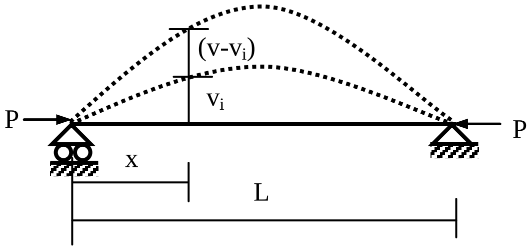
The critical load shown in Eq. (vii) is also called the buckling load or Euler load of the column, named after Leonhard Euler who was the first to derive it.

Euler's solution presents the buckling of column as a bifurcation problem; i.e., according to it the column would not deflect at all until it reaches the first critical load (= $\pi^2 EI/L^2$), where its deflection is arbitrary. After exceeding this load, the column returns to its un-deflected position until it reaches the second critical load (= $4\pi^2 EI/L^2$), and so on.



Obviously, the Euler solution is not consistent with the observed structural behavior of axially loaded columns. The discrepancy can be attributed to the assumptions in deriving the formula; i.e., that the column is perfectly straight, the applied load is concentric, the support condition is pin-pinned, the material follows Hooke's law and that there is no residual stress in the column (which is inappropriate for steel).

Effect of Initial Imperfection



$EI\,(v-v_i)'' = -Pv \Rightarrow EI\,v'' + Pv = EI\,v_i''$

$\Rightarrow v'' + \lambda^2 v = v_i''$ ……………(i) [where $\lambda^2 = P/EI$]

If $v_i(x) = v_{0i} \sin(\pi x/L)$

$v_i''(x) = -(\pi/L)^2 v_{0i} \sin(\pi x/L)$ ……………..(ii)

$\therefore$ (i) $\Rightarrow v'' + \lambda^2 v = -(\pi/L)^2 v_{0i} \sin(\pi x/L)$

$\Rightarrow v(x) = C_1 \cos\lambda x + C_2 \sin\lambda x - (\pi/L)^2/[\lambda^2-(\pi/L)^2]\, v_{0i} \sin(\pi x/L)$ ………………(iii)

$\therefore v(0) = 0 \Rightarrow C_1 = 0$ ………………(iv)

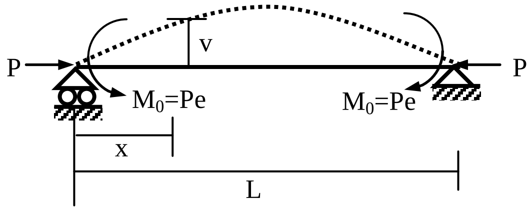
$v(L) = 0 \Rightarrow C_2 \sin(\lambda L) = 0 \Rightarrow C_2 = 0$ ………………(v)

$\Rightarrow v(x) = 1/[1-(\lambda L/\pi)^2]\, v_{0i} \sin(\pi x/L)$ ………………(vi)

$\lambda^2 = P/EI \Rightarrow v(x) = 1/[1-P/(\pi^2 EI/L^2)]\, v_{0i} \sin(\pi x/L) = v_{0i}/[1-P/P_{cr}] \sin(\pi x/L)$ ………..(vii)

$\therefore v(L/2) = v_{0i}/[1-P/P_{cr}]$ ……………..(viii)

Effect of Load Eccentricity (End Moments)



$EI\,v'' = -Pv - Pe$

$\Rightarrow EI\,v'' + Pv = -Pe$

$\Rightarrow v'' + \lambda^2 v = -\lambda^2 e$ ………..(i)

[where $\lambda^2 = P/EI$]

$\therefore$ (i) $\Rightarrow v(x) = C_1 \cos\lambda x + C_2 \sin\lambda x - e$ ……………(ii)

$\therefore v(0) = 0 \Rightarrow C_1 = e$ ……………(iii)

$v(L) = 0 \Rightarrow C_2 \sin(\lambda L) = e\,(1-\cos\lambda L) \Rightarrow C_2 = e\tan(\lambda L/2)$ ……………(iv)

$\Rightarrow v(x) = e\,[\cos\lambda x - 1 + \tan(\lambda L/2)\sin\lambda x]$ ……………(v)

$\therefore v(L/2) = e\,[\cos\lambda L/2 - 1 + \tan(\lambda L/2)\sin\lambda L/2]$

$= e\,[\sec\lambda L/2 - 1] = e\,[\sec\{(\pi/2)\sqrt{(P/P_{cr})}\}-1]$ …………..(vi)

(v) $\Rightarrow EIv''(x) = -EI\lambda^2 e\,[\cos\lambda x + \tan(\lambda L/2)\sin\lambda x]$

$\therefore M(x) = -M_0\,[\cos\lambda x + \tan(\lambda L/2)\sin\lambda x]$ ……………(vii)

$\therefore M(L/2)/M_0 = -[\sec(\lambda L/2)]$ ……………(viii)

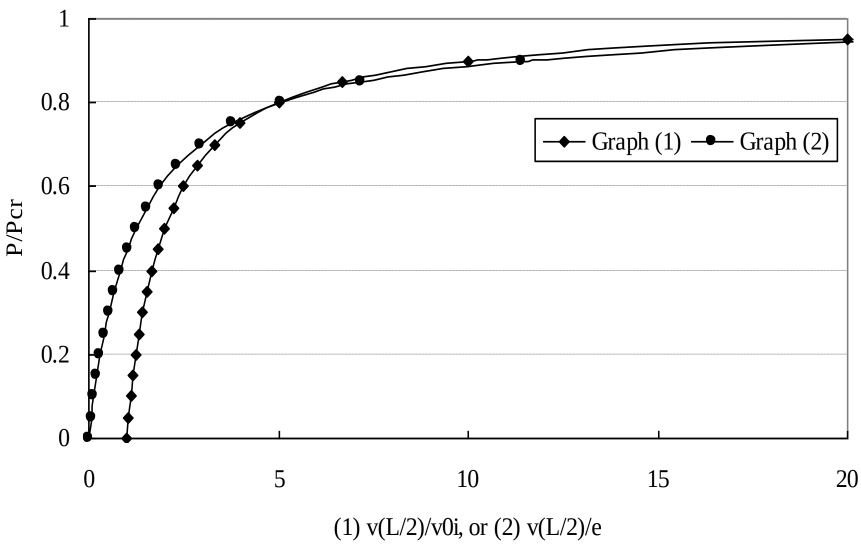


Fig. 1: Effect of (1) Column Imperfection and (2) Load Eccentricity

<u>Effect of Material Nonlinearity</u>

For materials with nonlinear stress-strain relationship, the critical load is $P_{cr} = \pi^2 E_t I/L^2 \Rightarrow \sigma_{cr} = \pi^2 E_t/\eta^2$

where E_t = Tangent modulus = $d\sigma/d\varepsilon$; i.e., slope of the stress-strain graph, η = Slenderness ratio

<u>Example</u>: Calculate σ_{cr} if $\sigma = 40\,\varepsilon - 20\,\varepsilon^2$ and η is (a) π, (b) 2π, (c) 4π.

$\sigma = 40\,\varepsilon - 20\,\varepsilon^2 \Rightarrow E_t = d\sigma/d\varepsilon = 40 - 40\,\varepsilon$

$\therefore 40\,\varepsilon - 20\,\varepsilon^2 = \pi^2(40 - 40\,\varepsilon)/\eta^2 \Rightarrow \varepsilon^2 - 2(1+\pi^2/\eta^2)\,\varepsilon + 2\pi^2/\eta^2 = 0 \Rightarrow \varepsilon = (1+\pi^2/\eta^2) - \sqrt{(1+\pi^4/\eta^4)}$ ……(1)

$\therefore$ (a) $\eta = \pi \Rightarrow \varepsilon = 0.586$, $\sigma = 16.57$, (b) $\eta = 2\pi \Rightarrow \varepsilon = 0.219$, $\sigma = 7.81$, (c) $\eta = 4\pi \Rightarrow \varepsilon = 0.061$, $\sigma = 2.35$

If $\sigma = 40\,\varepsilon$, $E_t = 40$ and the corresponding strains and stresses are

(a) $\eta = \pi \Rightarrow \varepsilon = 1$, $\sigma = 40$, (b) $\eta = 2\pi \Rightarrow \varepsilon = 0.25$, $\sigma = 10$, (c) $\eta = 4\pi \Rightarrow \varepsilon = 0.0625$, $\sigma = 2.5$

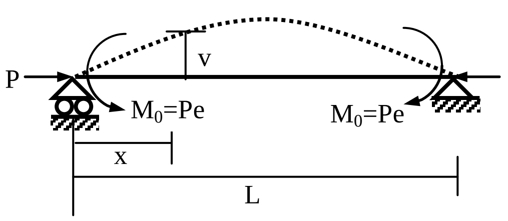
<u>Elasto-plastic Material Property</u>

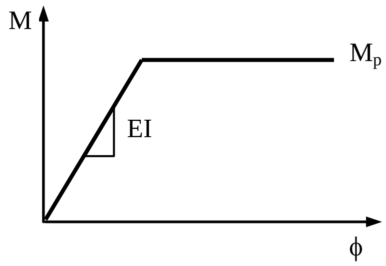
The material property of an eccentrically loaded beam-column is elasto-plastic; i.e.,

Bending moment $M = EI\,v''$, with an upper limit of M_p

$\Rightarrow -Pv - Pe = EI\,v''$, for the elastic range ..………..(i)

and $-Pv - Pe = -M_p$, for the plastic range ..………..(ii)





$\therefore v(L/2) = e\,[\sec\{(\pi/2)\sqrt{(P/P_{cr})}\}-1]$, for the elastic range …………(iii)

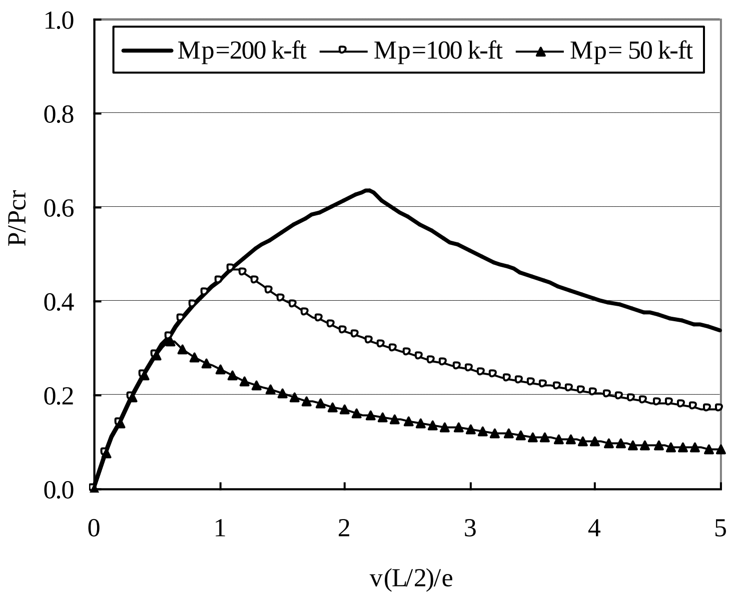
$v(L/2) = M_p/P - e$, for the plastic range …..……....(iv)

<u>Example</u>: If $EI = 40000$ k-ft^2, $L = 20$ ft, $e = 0.10$ ft, M_p is (a) 200, (b) 100, (c) 50 k-ft.

$\therefore P_{cr} = \pi^2 EI/L^2 = 987$ kips …………(1)

$\therefore$ For the elastic range, $v(L/2)/e = \sec\{\sqrt{(P/400)}\} - 1$ …………(2)

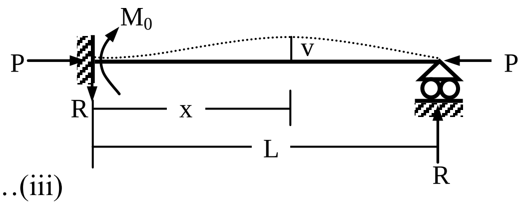
and for the plastic range, $v(L/2)/e$ is (a) $2000/P - 1$, (b) $1000/P - 1$, (c) $500/P - 1$ …………(3)



Effect of Material Nonlinearity

Effect of End Conditions

(a) Fixed-Hinged column



$EI\, v'' = M_0 - Rx - Pv$

$\Rightarrow EI\, v'' + Pv = M_0 - (M_0/L)\, x$ ……….(i)

$\therefore v(x) = C_1 \cos \lambda x + C_2 \sin \lambda x + [M_0 - (M_0/L)x]/P$ ……….(ii)

[where $\lambda^2 = P/EI$]

$\therefore v(0) = 0 \Rightarrow C_1 = -M_0/P$ ……….(iii)

$v'(0) = 0 \Rightarrow C_2 = M_0/P\lambda L$ ……….(iv)

$\therefore v(x) = (M_0/P)\,[-\cos \lambda x + (1/\lambda L) \sin \lambda x + 1 - x/L]$ ……….(v)

$\therefore v(L) = 0 \Rightarrow \cos \lambda L = (1/\lambda L) \sin \lambda L \Rightarrow \tan \lambda L = \lambda L$ ……….(vi)

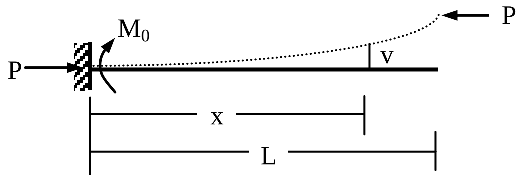
$\therefore \lambda L = 4.49341$ ……….(vii)

$\therefore\ P = P_{cr} = EI\lambda^2 = 20.19\, EI/L^2 = 2.046\, (\pi^2 EI/L^2) \cong \pi^2 EI/(0.7L)^2$ ……….(viii)

(b) For a Fixed-Fixed column, similarly derive

$P = P_{cr} = 4\,(\pi^2 EI/L^2) = \pi^2 EI/(0.5L)^2$ ……….(ix)

(c) Fixed-Free (Cantilever) column



$EI\, v'' = M_0 - Pv$

$\Rightarrow EI\, v'' + Pv = M_0$ ……….(x)

$\therefore v(x) = C_1 \cos \lambda x + C_2 \sin \lambda x + M_0/P$ ……….(xi)

[where $\lambda^2 = P/EI$]

$\therefore v(0) = 0 \Rightarrow C_1 = -M_0/P$ ……….(xii)

$v'(0) = 0 \Rightarrow C_2 = 0$ ……….(xiii)

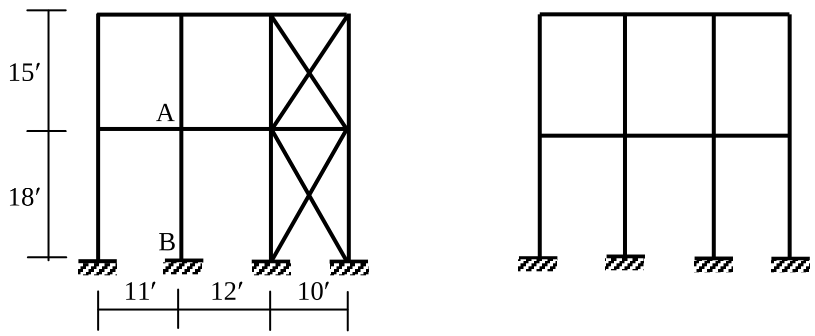
$\therefore v(x) = (M_0/P)\,(1 - \cos \lambda x)$ ……….(xiv)

$\therefore v''(L) = 0 \Rightarrow \lambda^2 \cos \lambda L = 0 \Rightarrow \lambda L = \pi/2$ ……….(xv)

$\therefore P = P_{cr} = EI\lambda^2 = \pi^2 EI/4L^2 = \pi^2 EI/(2L)^2$ ……….(xvi)

In general, for any support condition, $P_{cr} = \pi^2 EI/L_e^2 = \pi^2 EI/(kL)^2$ ……….(xvii)
where L_e = Effective length of the column = kL, k depends on support conditions at the two ends; i.e., the relative stiffness of the compression members and flexural members at the two ends.

Example: Calculate the buckling load in column AB if the frame is (i) braced, (ii) unbraced.
[Given: EI = constant = 40,000 k-ft^2]

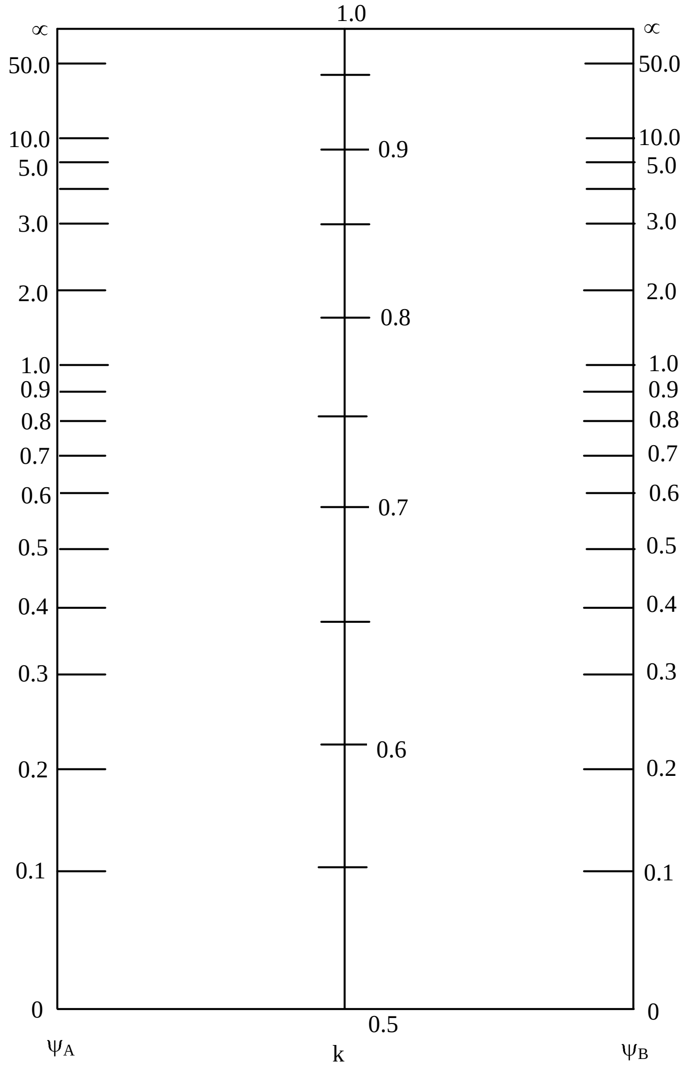


(a) Braced (b) Unbraced

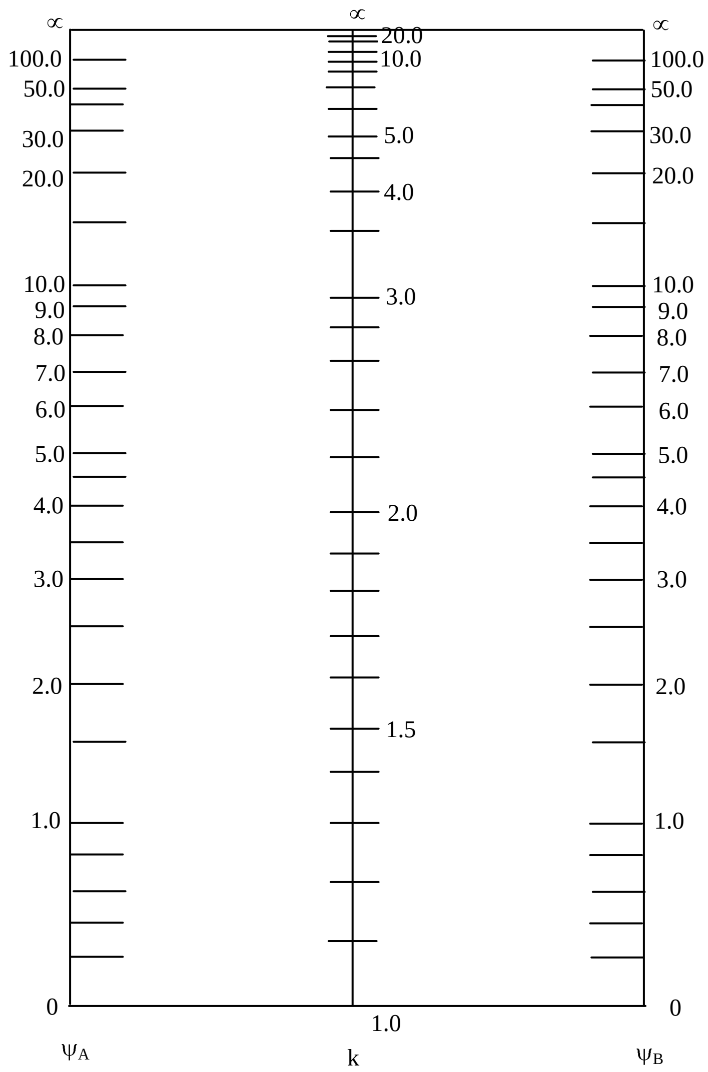
$\psi_A = (EI/15+EI/18)/(EI/11+EI/12) = 0.122/0.174 = 0.70$, $\psi_B = 0$

$\therefore$ For braced frame, k = 0.61, $P_{cr} = \pi^2 EI/(kL)^2 = \pi^2 \times 40000/(0.61\times 18)^2 = 3275$ kips

For unbraced frame, k = 1.12, $P_{cr} = \pi^2 \times 40000/(1.12\times 18)^2 = 971$ kips



Braced Frames



Unbraced Frames

ψ = Ratio of $\sum EI/L$ of compression members to $\sum EI/L$ of flexural members in a plane at one end of a compression member.
k = Effective length factor.

Alignment Charts for Effective Length Factors k

Design Concept of Axially Loaded Members

For structural members under compression, the AISC-ASD (AISC ≡ American Institute of Steel Construction, ASD ≡ Allowable Stress Design) guidelines recommend the following equations

Slenderness Ratio, $\eta = L_e/r_{min}$, and $\eta_c = \pi\sqrt{(2E/f_y)}$

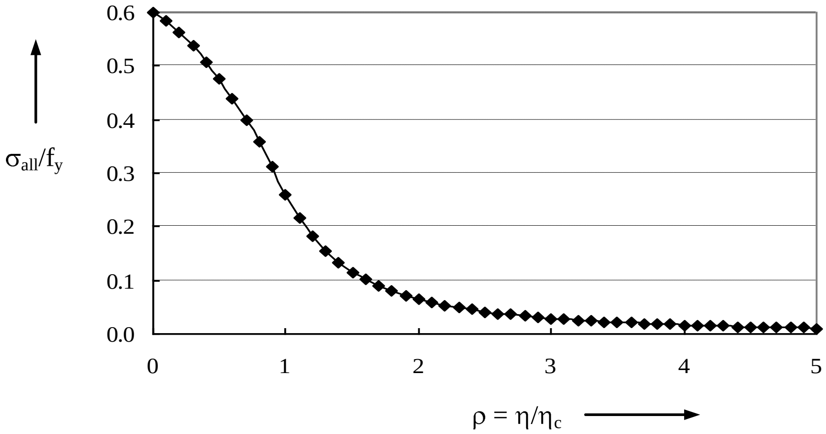
If $\eta \le \eta_c$, $\sigma_{all} = f_y\,[1-0.5\,(\eta/\eta_c)^2]/FS$,

where FS = Factor of safety = $[5/3 + 3/8\,(\eta/\eta_c) - 1/8\,(\eta/\eta_c)^3]$ ………………(i)

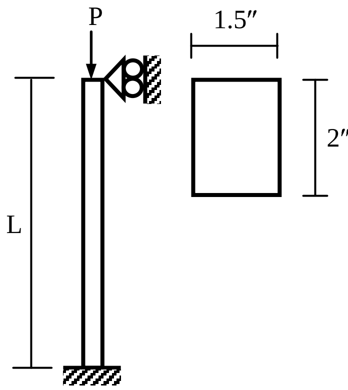
If $\eta > \eta_c$, $\sigma_{all} = (\pi^2 E/\eta^2)/FS$, where FS = Factor of safety = 23/12 = 1.92 ……………...(ii)

Here E = Modulus of elasticity, f_y = Yield strength, σ_{all} = Allowable compressive stress
A = Cross-sectional area, L_e = Effective length of member, r_{min} = Minimum radius of gyration

The AISC-ASD column design curve is shown in the figure below

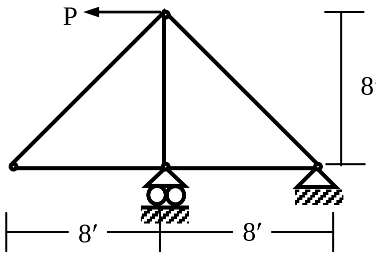


Example: Calculate P_{allow} for the column below using the AISC-ASD Method if
(a) L = 5′, (2) L = 10′ [Given: E = 29000 ksi, f_y = 40 ksi]

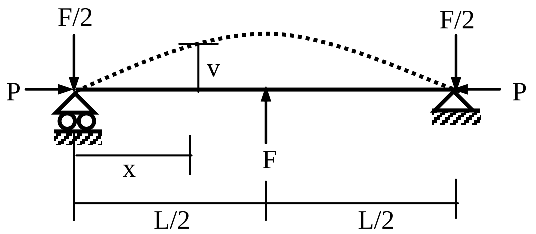


$\eta_c = \pi\sqrt{(2E/f_y)} = \pi\sqrt{(2\times29000/40)} = 119.63$
k = 0.7 [Fixed-Hinged], $r_{min} = 1.5''/\sqrt{(12)} = 0.433''$
(a) L = 5′ = 60″ $\Rightarrow \eta = kL/r_{min} = 0.7\times60/0.433 = 96.99 < \eta_c$
$\sigma_{cr} = f_y\,[1 - 0.5(\eta/\eta_c)^2] = 40\times[1 - 0.5\times(96.99/119.63)^2] = 26.85$ ksi
$FS = 5/3 + 0.375(\eta/\eta_c) - 0.125\,(\eta/\eta_c)^3 = 1.90$
$\therefore \sigma_{all} = \sigma_{cr}/FS = 26.85/1.90 = 14.10$ ksi $\Rightarrow P_{all} = 14.10 \times 3.0 = 42.30$ k
(b) L = 10′ = 120″ $\Rightarrow \eta = kL/r_{min} = 0.7\times120/0.433 = 193.98 > \eta_c$
$\sigma_{cr} = \pi^2E/\eta^2 = \pi^2\times29000/193.98^2 = 7.61$ ksi, and FS = 1.92
$\therefore \sigma_{all} = \sigma_{cr}/FS = 7.61/1.92 = 3.96$ ksi $\Rightarrow P_{all} = 3.96 \times 3.0 = 11.88$ k

Problem: Check the adequacy of the truss against buckling using AISC-ASD criteria, if $P = 10^k$
[Given: A = 1.2 in^2, r_{min} = 1″, E = 29000 ksi, f_y = 40 ksi for all members].



Moment Magnification



<u>Concentrated Load at the Midspan of a Simply Supported Beam</u>

EI $v'' = -Pv - Fx/2$ $\quad (x < L/2)$

$\Rightarrow$ EI $v'' + Pv = -Fx/2$ ………….(i)

$\therefore$ Using $\lambda^2 = P/EI$

$v(x) = C_1 \cos \lambda x + C_2 \sin \lambda x - Fx/(2P)$ ………….(ii)

$\therefore v(0) = 0 \Rightarrow C_1 = 0$ …………..(iii)

$v'(L/2) = 0 \Rightarrow C_2 \lambda \cos(\lambda L/2) = F/(2P) \Rightarrow C_2 = F/(2P\lambda \cos(\lambda L))$ …………….(iv)

$\Rightarrow v(x) = FL/(2P)\,[\sin(\lambda x)/\{(\lambda L)\cos(\lambda L/2)\} - x/L]$ ..…………...(v)

$\Rightarrow M(x) = EI\, v''(x) = (-\lambda^2 EI)\, FL/(4P)\,[\sin(\lambda x)/\{(\lambda L/2)\cos(\lambda L/2)\}]$

$= -(FL/4)\,[\sin(\lambda x)/\{(\lambda L/2)\cos(\lambda L/2)\}]$ …………….(vi) [Using $\lambda^2 = P/EI$]

$\therefore M(L/2) = -(FL/4)\,[\tan(\lambda L/2)/(\lambda L/2)]$ …….…….(vii)

$\therefore$ Moment magnification factor = $[\tan(\lambda L/2)/(\lambda L/2)]$ ..…..……..(viii)

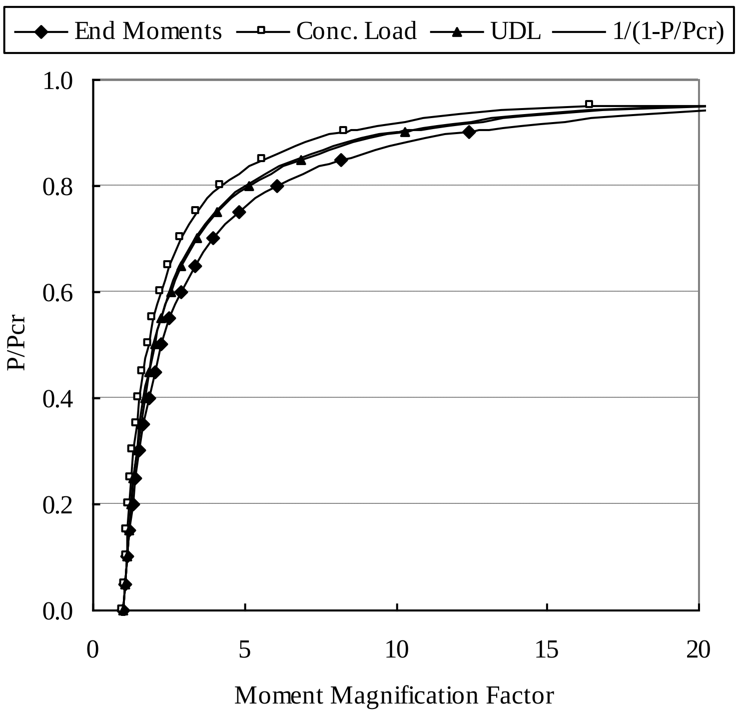
Similarly, moment magnification factor for simply supported beam subjected to end moments only

$= [\sec(\lambda L/2)]$, as derived earlier …….……..(ix)

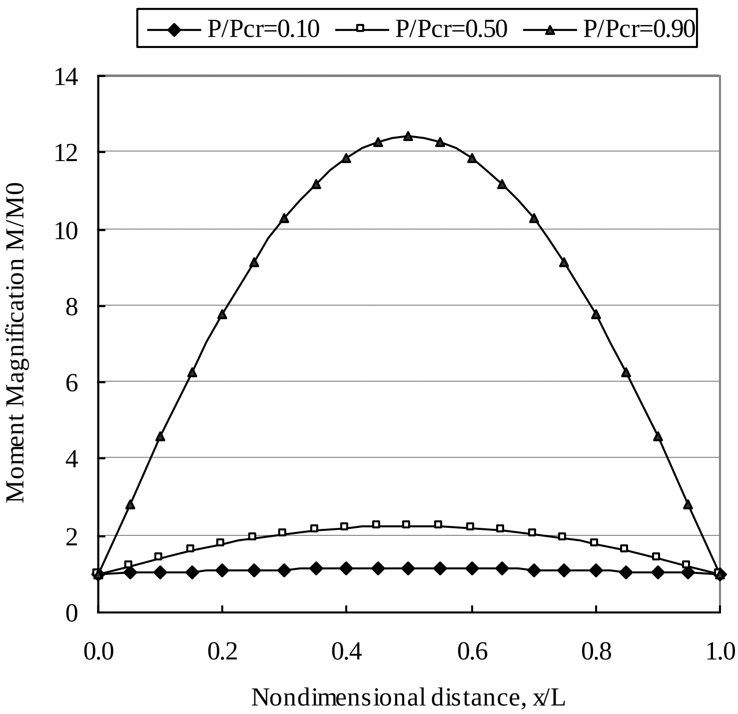
Also, moment magnification factor for simply supported beam under UDL

$= 2\,[\sec(\lambda L/2) - 1]/(\lambda L/2)^2$ …..……….(x)

The moment magnification factor according to AISC code = $1/(1 - P/P_{cr})$ …….……..(xi)



Moment Magnification Factors for Various Loads



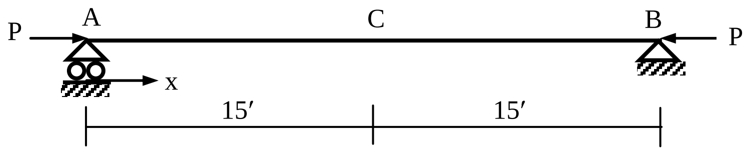
Magnification of BMD for End Moments

Buckling of Columns

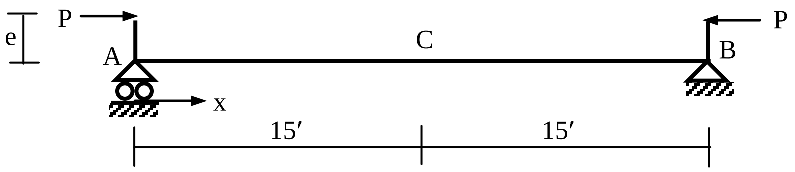
1. State the effects of (i) initial imperfection, (ii) load eccentricity, (iii) material nonlinearity, (iv) residual stresses on the critical buckling load or buckling characteristics of a slender column under compression.

2. Draw the axial load vs. lateral deflection curve of an ideal column according to Euler's formulation. Explain why (i) Real columns start bending from the beginning of (axial) loading, (ii) Real columns fail at axial loads smaller than the Euler load.

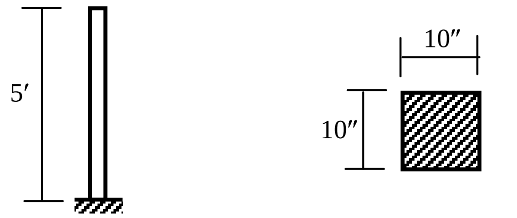
3. The beam ACB shown below has an initial deflected shape of $v_i(x) = v_{0i} \sin(\pi x/L)$. If the deflection at C for P = 100 kips is 1″, calculate the value of v_{0i} and the deflection at C for P = 200 kips [Given: EI = 4×10^6 k-in^2].



4. The beam ACB shown below is subjected to compressive loads (P) applied at both ends at an eccentricity of 'e'. If the deflection at C for P = 100 kips is 1″, calculate the value of 'e' and the deflection at C for P = 200 kips [Given: EI = 4×10^6 k-in^2].

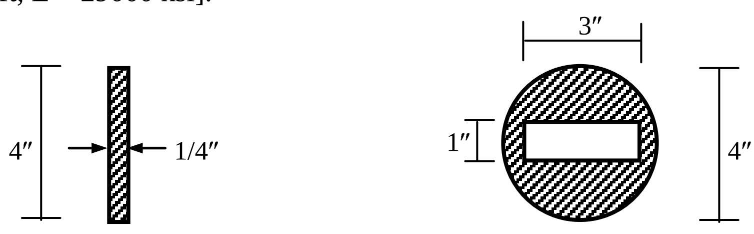


5. A 5-ft long cantilever column has a 10″×10″ cross-section as shown below and is made of a nonlinear material with stress-strain relationship given by $\sigma = 4(1-e^{-100\varepsilon})$, where σ is the stress (ksi) and ε is the strain. Calculate the critical load for the column.

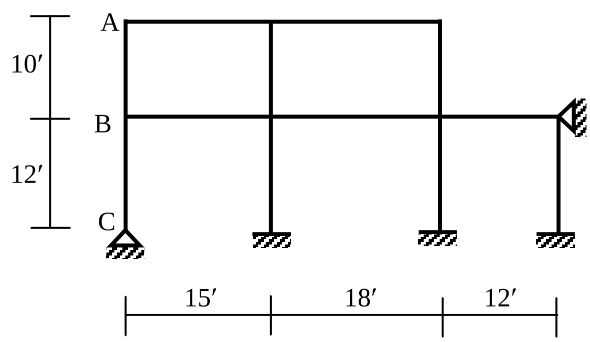


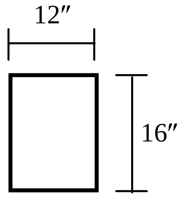
Cantilever Column
Cross-section

6. The figure below shows cross-sectional areas of two columns. Calculate the critical buckling loads of the columns if (i) One end is fixed and the other is free, (ii) Both ends are fixed [Given: Length of columns = 10 ft, E = 29000 ksi].



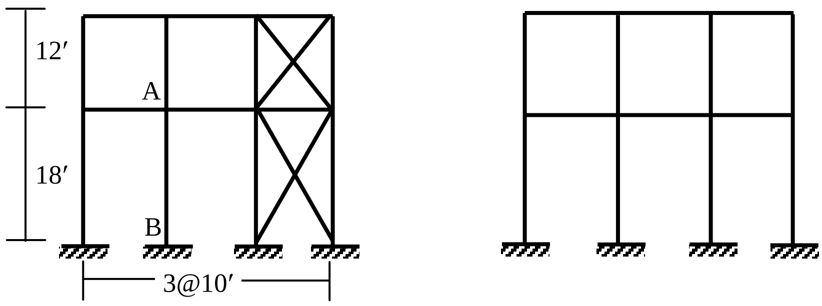
7. Calculate the Euler loads for columns AB and BC in the frame shown below [Given: E = 3000 ksi, EI = constant].



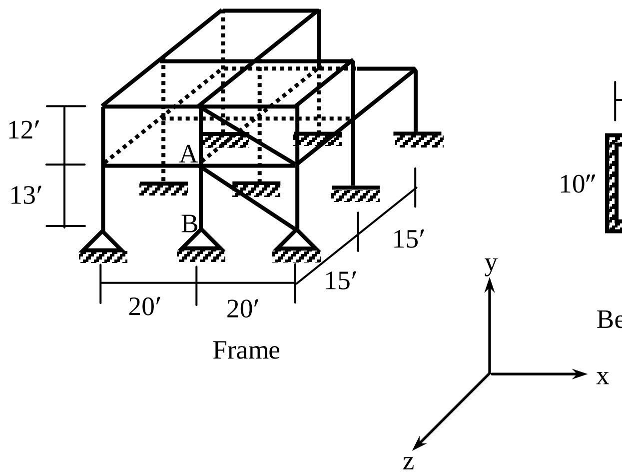


Cross-section

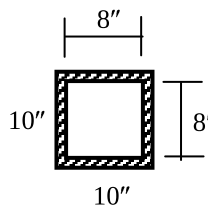
8. Calculate the buckling load in column AB if the frame is (i) braced, (ii) unbraced [Given: EI = constant = 40,000 k-ft^2].



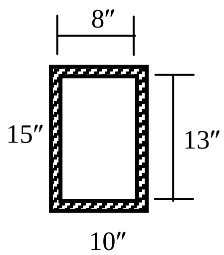
9. In the frame shown below, calculate the effective length factor of column AB about x and z-axis and determine the minimum allowable compressive force on the column according to AISC-ASD criteria [Given: E = 30×10^3 ksi, f_y = 40 ksi, member sections are shown below].



Frame



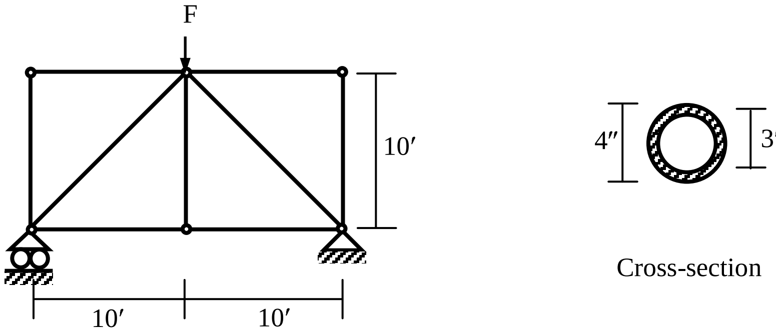
Beam Section



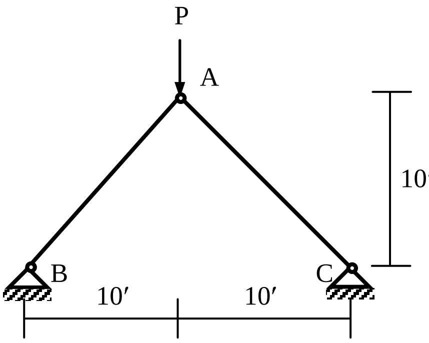
Column Section

10. Draw the AISC-ASD design curve for steel columns with E = 29000 ksi and yield strength = 36 ksi. Show all the relevant details like η_c, σ_{allow} at η = 0 and $\eta = \eta_c$ on the graph.

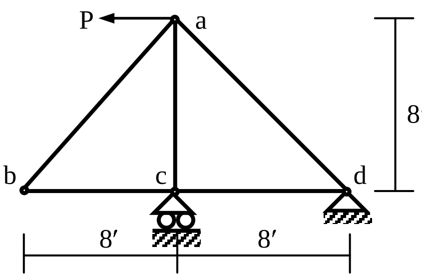
11. Calculate the allowable value of F for the truss shown below using the AISC-ASD criteria [Given: The truss members are hollow circular tubes of 4″ outside and 3″ inside diameter, E = 29000 ksi, f_y = 50 ksi for all members].



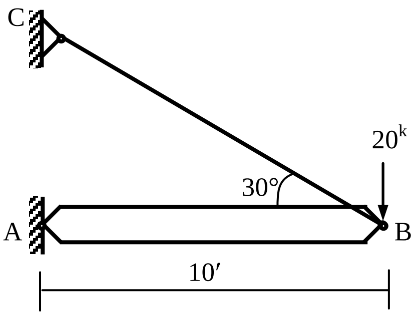
12. In the structure shown below, check the adequacy of the member AB using the AISC-ASD criteria for buckling, if P = 100^k [Given: AB is a solid circular tube of 4″ diameter, yield strength f_y = 40 ksi, modulus of elasticity E = 29000 ksi].



13. In the structure shown below, check the adequacy of the truss against buckling using the AISC-ASD criteria, if P = 10^k [Given: Area A = 1.2 in^2, r_{min} = 1″, Modulus of Elasticity E = 29000 ksi, yield strength = 40 ksi, for all members].

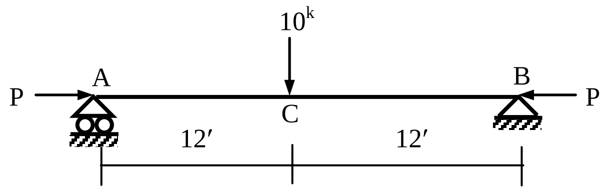


14. Check the adequacy of the structure shown below against buckling if P = 20^k. Use a factor of safety of 2 and assume ends A, B and C are all hinged [Given: AB is a solid tube of 3″ diameter, BC is a cable of 1/8″ diameter, modulus of elasticity E = 29000 ksi].

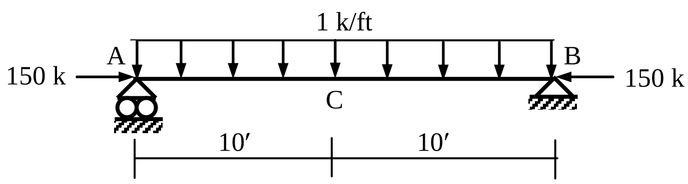


15. What is 'moment magnification factor'? What are the steps involved in calculating the moment magnification factor of a typical column in a multi-storied building?

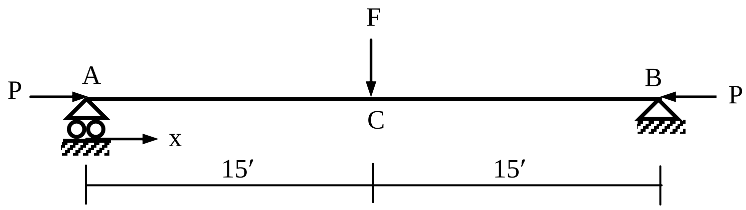
16. Using the AISC moment magnification factor (with C_m = 0.85), calculate the bending moment at the mid-span C of the simply supported beam shown below if
    (i) P = 0, and (ii) P = 300^k [Given: E = 3000 ksi, I = 1728 in^4].



17. For the simply supported beam shown below, calculate
    (i) the allowable axial load using the AISC-ASD method
    (ii) the bending moment at the mid-span C
    [Given: E = 29000 ksi, f_y = 60 ksi, A = 10 in^2, I_{min} = 90 in^4].

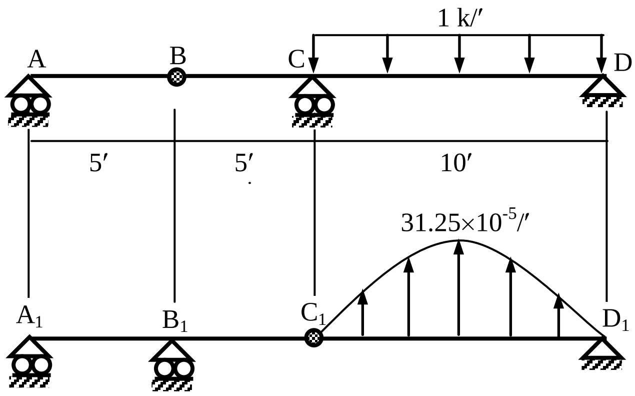


18. For the beam ACB shown below, compare the deflections at C for P = 0 and P = 200 kips, if
    (i) F = 0 but the beam has an initial deflected shape of $v_i(x) = v_{0i} \sin(\pi x/L)$, where v_{0i} = 1 in,
    (ii) F = 10 kips and the beam is perfectly straight initially [Given: EI = 4×10^6 k-in^2].



Solution of Midterm Question

1.



$EI = 40\times10^{3}$ k-ft^2

Conjugate Beam loaded by M/EI

In the conjugate beam, BM at $C_1 = 0 \Rightarrow -(2/3)\times31.25\times10^{-5}\times10\times5 - 10\times R_{D1} = 0$

$\Rightarrow R_{D1} = -104.17\times10^{-5}$ rad ……………….(1)

$\therefore$ In the separated beam C_1D_1, $R_{C1} = -104.17\times10^{-5}$ rad

$\therefore$ In the separated beam $A_1B_1C_1$, $R'_{C1} = 104.17\times10^{-5}$ rad ……………….(2)

$\Rightarrow$ BM at $B_1 = 104.17\times10^{-5}\times5 = 520.83\times10^{-5}$ ft

$\Rightarrow v_B = 520.83\times10^{-5}$ ft $= 0.0625''$ $\uparrow$ ……………….(3)

2. <u>Moment-Area Theorem</u>:

Using 1st theorem between C and D $\Rightarrow \theta_D - \theta_C = (2/3)\times31.25\times10^{-5}\times10 = 208.33\times10^{-5}$ rad .....……(1)

Using 2nd theorem between C and D $\Rightarrow (x_D - x_C)\theta_D - v_D + v_C = (2/3)\times31.25\times10^{-5}\times10\times5$

$\Rightarrow 10\,\theta_D - 0 + 0 = 104.17\times10^{-4} \Rightarrow \theta_D = 104.17\times10^{-5}$ rad …..….(2)

$\therefore (1) \Rightarrow \theta_C = -\,104.17\times10^{-5}$ rad …..….(3)

Using 1st theorem between $B^{(+)}$ and C $\Rightarrow \theta_C - \theta_{B(+)} = 0$ ………….(4)

Using 2nd theorem between B and C $\Rightarrow (x_C - x_B)\theta_C - v_C + v_B = 0$

$\Rightarrow 5\,\theta_C - 0 + v_B = 0$

$\Rightarrow v_B = -\,5\theta_C = 520.83\times10^{-5}$ ft $= 0.0625''$ $\uparrow$ ……....…(5)

<u>Singularity Function</u>:

$EIv^{iv}(x) \cong w(x) = C_\theta\langle x-5\rangle^{-3}_{*} + R_C\langle x-10\rangle^{-1}_{*} - 1\langle x-10\rangle^{0}$

$EIv'''(x) \cong V(x) = C_\theta\langle x-5\rangle^{-2}_{*} + R_C\langle x-10\rangle^{0} - 1\langle x-10\rangle^{1} + C_1$ ……....…(1)

$EIv''(x) \cong M(x) = C_\theta\langle x-5\rangle^{-1}_{*} + R_C\langle x-10\rangle^{1} - 1/2\langle x-10\rangle^{2} + C_1\,x + C_2$ ……....…(2)

$EIv'(x) = S(x) \cong C_\theta\langle x-5\rangle^{0} + R_C/2\langle x-10\rangle^{2} - 1/6\langle x-10\rangle^{3} + C_1x^2/2 + C_2\,x + C_3$ …...……(3)

$EIv(x) = D(x) \cong C_\theta\langle x-5\rangle^{1} + R_C/6\langle x-10\rangle^{3} - 1/24\langle x-10\rangle^{4} + C_1x^3/6 + C_2x^2/2 + C_3\,x + C_4$ ……....…(4)

$M(0) = 0 \Rightarrow C_2 = 0$ ………….(5)

$M(5) = 0 \Rightarrow 5C_1 = 0 \Rightarrow C_1 = 0$ ………….(6)

$M(20) = 0 \Rightarrow 10\,R_C - 50 = 0 \Rightarrow R_C = 5$ ………….(7)

$D(0) = 0 \Rightarrow C_4 = 0$ ………….(8)

$D(10) = 0 \Rightarrow 5C_\theta + 10\,C_3 = 0$ ………….(9)

$D(20) = 0 \Rightarrow 15\,C_\theta + 833.33 - 416.67 + 20\,C_3 = 0 \Rightarrow 15\,C_\theta + 20\,C_3 = -\,416.67$ ………...(10)

Solving (9), (10) $\Rightarrow C_3 = 41.67$, $C_\theta = -83.33$ ..……….(11)

$\therefore EIv(x) = D(x) \cong -83.33\langle x-5\rangle^{1} + 0.8333\langle x-10\rangle^{3} - 0.04167\langle x-10\rangle^{3} + 41.67\,x$ ..……….(12)

$v(5) = D(5)/EI \cong 41.67\times5/(40\times10^{3}) = 520.83\times10^{-5}$ ft $= 0.0625''$ $\uparrow$ ……....…(13)

3. $L_e = 2\times5\times12 = 120$ in, $A = 10\times10 = 100$ in^2, $r_{min} = 10/\sqrt{12} = 2.887''$ ………………(1)

$\Rightarrow \eta = L_e/r_{min} = 41.57$

$\sigma = 4(1 - e^{-100\varepsilon}) \Rightarrow E_t = d\sigma/d\varepsilon = 400\, e^{-100\varepsilon}$

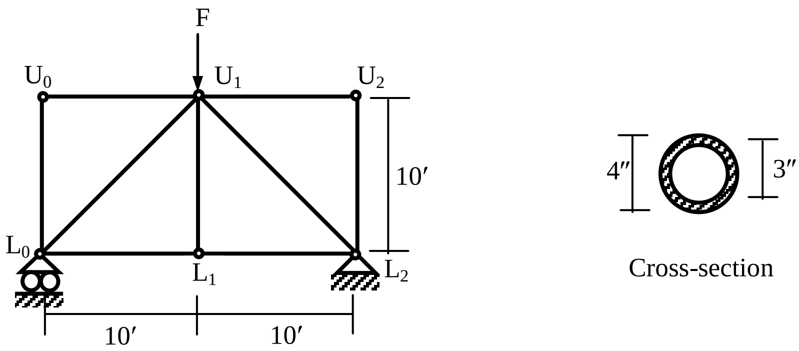
$\therefore 4(1 - e^{-100\varepsilon}) = \pi^2\times400\, e^{-100\varepsilon}/\eta^2 = \pi^2\times400\, e^{-100\varepsilon}/(41.57)^2 \Rightarrow 1 - e^{-100\varepsilon} = 0.571\, e^{-100\varepsilon}$

$\Rightarrow e^{-100\varepsilon} = 1/1.571 \Rightarrow e^{100\varepsilon} = 1.571 \Rightarrow \varepsilon = 4.52\times10^{-3}$ ………………(2)

$\therefore \sigma_{cr} = 4(1 - e^{-100\varepsilon}) = 4(1 - e^{-0.452}) = 1.454$ ksi

$\Rightarrow P_{cr} = 1.454\times100 = 145.4$ kips ………………(3)

4.



Cross-section

The only compression members in the truss are U_1L_0 and U_1L_2

Each of them is 14.14′ long and under compressive force of 0.707F

$A = \pi(4^2-3^2)/4 = 5.50$ in^2, $I_{min} = \pi(4^4-3^4)/64 = 8.59$ in$^4 \Rightarrow r_{min} = \sqrt{(8.59/5.50)} = 1.25''$

$\Rightarrow \eta = L_e/r_{min} = 14.14\times12\,/1.25 = 135.76$ and $\eta_c = \pi\sqrt{(2E/f_y)} = \pi\sqrt{(2\times29000/50)} = 107.00$

$\therefore \eta > \eta_c \Rightarrow \sigma_{all} = (\pi^2 E/\eta^2)/FS = (\pi^2\times29000/135.76^2)/1.92 = 8.09$ ksi

$\therefore P_{all} = \sigma_{all} A = 8.09 \times 5.50 = 44.46$ kips

$\therefore 0.707F_{all} = 44.46 \Rightarrow F_{all} = 62.88$ kips