Chapter 5

Response Analysis for Multi Support Earthquake Excitation

5.1 Introduction

It is very important to perform the dynamic analysis for the structure subjected to random/dynamic loadings. The dynamic analysis of structures mainly involves the response spectrum analysis and time history analysis. In some of the structures having very large spans, the effects of ground motion at different supports may be different and in such cases it is necessary to perform the time history analysis considering the effects of time delay of earthquake ground motions. This chapter deals with the derivation of equations of motion for single and multi degree of freedom systems subjected to single and multi support earthquake excitations. Further, a step by step procedure is explained to calculate the numerical response by using state space method.

5.2. Equations of Motion for Single Degree of Freedom (SDOF) System subjected to Earthquake Excitation

Consider a SDOF system as shown in Figure 5.1, which is subjected to an earthquake ground motion. Four types of forces will be acting on the mass as follows,

i) Inertial force (F_I)
ii) Stiffness force (F_S)
iii) Damping force (F_D)
iv) External force (F_E)

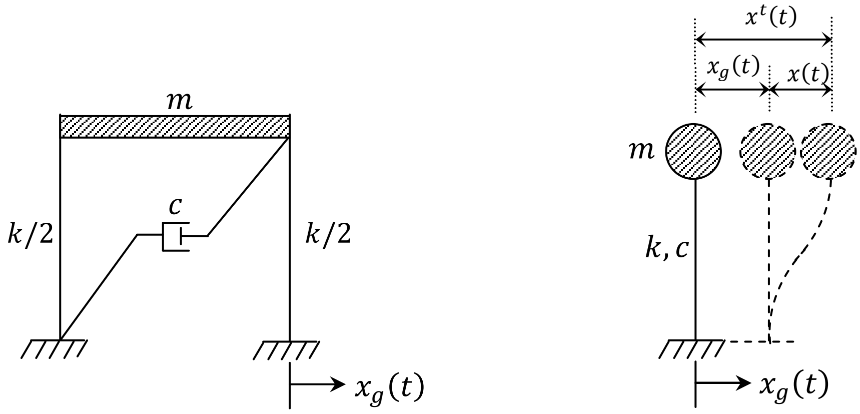


Figure 5.1 (a) SDOF system (b) Lumped mass model of SDOF

In Figure 5.1 (a), the SDOF system is represented by a mass at the top of a column. The rotation and vertical deflection at the end of the columns are ignored. Also, the floor is assumed to be axially rigid. Figure 5.1 (b) represents the lumped mass distribution of SDOF system (Chopra, 2007; Clough and Penzien, 1993; Datta, 2010).

From Newton's law, where the sum of forces is equal to the mass time acceleration,

$$\sum F = F_S + F_D + F_I = F_E \tag{5.1}$$

Stiffness force (F_S): This force acts on the floor when there is a lateral displacement of the mass. For a linear system, this force is directly proportional to the relative displacement of the top and bottom ends of the column.
Damping force (F_D): This force acts on the floor when there is a relative lateral velocity between the mass and the ground. For a linear viscous damping, this force is directly proportionally to the velocity and the constant of proportionality is the damping coefficient.

External force (F_E): This force is an external force applied to the system.

Inertial force (F_I): This represents the inertial force due to the acceleration of the floor.

As shown in Figure 5.1 (b), $x^t(t)$ is the absolute displacement of mass and $x_g(t)$ is the absolute displacement of the ground. The relative displacement between the mass and the ground is denoted by $x(t)$.

$$\begin{aligned} x(t) &= x^t(t) - x_g(t) \\ \dot{x}(t) &= \dot{x}^t(t) - \dot{x}_g(t) \\ \ddot{x}(t) &= \ddot{x}^t(t) - \ddot{x}_g(t) \end{aligned} \tag{5.2}$$

Hence, the stiffness force,

$$F_S = -k\,x(t) = -k\{\,x^t(t) - x_g(t)\} \tag{5.3}$$

Similarly, the damping force,

$$F_D = -c\,\dot{x}(t) = -c\,\{\dot{x}^t(t) - \dot{x}_g(t)\} \tag{5.4}$$

The inertial force is mass times the absolute acceleration.

Hence,

$$F_I = m\,\ddot{x}^t(t) = m\{\ddot{x}(t) + \ddot{x}_g(t)\} \tag{5.5}$$

Now, Rewriting Equation (5.1) in form of above equations,

$$-k\,x(t) - -c\,\dot{x}(t) = m\{\ddot{x}(t) + \ddot{x}_g(t)\} \tag{5.6}$$

Therefore,

$$m\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = -\,m\,\ddot{x}_g(t) \tag{5.7}$$

Equation (5.7) is defined as the equation of motion for the SDOF system, subjected to ground acceleration $\ddot{x}_g(t)$.

where,
m = mass of the system
c = damping coefficient of the damper system
k = stiffness of the structural system
x = relative displacement of mass with respect to ground/support
$\dot{x}$ = relative velocity of mass with respect to ground/support
$\ddot{x}$ = relative acceleration of mass with respect to ground/support
$\ddot{x}_g$ = ground acceleration
Now, Substituting Equation (5.2) in Equation (5.7), and rearranging the terms,

$$m\ddot{x}^t(t) + c\,\dot{x}^t(t) + k\,x^t(t) = c\,\dot{x}_g(t) + k\,x_g(t) \tag{5.8}$$

Equation (5.8) is defined as the equation of motion for the SDOF system, subjected to ground acceleration $\ddot{x}_g(t)$ in terms of absolute (total) motion of the mass.

5.3 Response of SDOF System: Solution by State Space Method

The equation of motion for SDOF system as derived earlier in Equation (5.7) may be rewritten as,

$$m\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = F_E(t) \tag{5.9}$$

where, $\mathrm{F_E}$ is the external force. Now, dividing Equation (5.9) with mass 'm',

$$\ddot{x}(t) + \frac{c}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{F_E(t)}{m} \tag{5.10}$$

Replacing $c = 2\xi m\omega_0$ and $\frac{k}{m} = \omega_0{}^2$ in above equation,

Thus,

$$\ddot{x}(t) + 2\xi\omega_0\,\dot{x}(t) + \omega_0{}^2\,x(t) = \frac{F_E(t)}{m} \tag{5.11}$$

State space method analyzes the response of the system using both the displacement and velocity as independent variables and these variables are called states. The two independent response variables are expressed as state vector, $\boldsymbol{z}$ which can be written as, (Hart and Wong, 2000)

$$\boldsymbol{z}(t) = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} \tag{5.12}$$

Further, Equation (5.11) can be written in matrix form as follows (Hart and Wong, 2000),

$$\dot{\boldsymbol{z}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0{}^2 & -2\xi\omega_0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_E(t)/m \end{Bmatrix} \tag{5.13}$$

Simplifying the above equation by substituting the following equations,

$$-\omega_0{}^2 = -\frac{k}{m} = -m^{-1}k \text{ and } -2\xi\omega_0 = -2\xi\left(\frac{c}{2\xi m}\right) = -m^{-1}c \tag{5.14}$$

Hence, Equation (5.13) will be,

$$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} 0 & 1 \\ -m^{-1}k & -m^{-1}c \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_E(t)/m \end{Bmatrix} \tag{5.15}$$

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\,\boldsymbol{z}(t) + \boldsymbol{F}(t) \tag{5.16}$$

Equation (5.16) is the 1st order linear matrix differential equation of motion and is called as continuous state space equation of motion.
In general, the solution for any time $t \geq t_0$, where 't_0' represents the time when the initial displacement and velocity are given , can be written as, (Hart and Wong, 2000)

$$\boldsymbol{z}(t) = \boldsymbol{e}^{\boldsymbol{A}(t-t_0)}\boldsymbol{z}(t_0) + \boldsymbol{e}^{\boldsymbol{A}t} \int_{t_0}^{t} \boldsymbol{e}^{-\boldsymbol{A}s}\,\boldsymbol{F}(s)ds \tag{5.17}$$

In the above equation, the matrix $\boldsymbol{e}^{\boldsymbol{A}t}$ is called state transition matrix and has the same dimension as '$\boldsymbol{A}$' matrix. If the initial conditions are given at time equal to zero (i.e. $t_0 = 0$), then

$$\boldsymbol{z}(t) = \boldsymbol{e}^{\boldsymbol{A}t}\mathbf{z}_0 + \int_{0}^{t} \boldsymbol{e}^{\boldsymbol{A}(t-s)}\,\boldsymbol{F}(s)ds \tag{5.18}$$

In the above Equation (5.18), the 1st part is the homogeneous solution with initial condition taken into considerations and 2nd part is the particular solution which is expressed in terms of time-integration of forcing function.

where, $$\boldsymbol{z}_o = \boldsymbol{z}(0) = \begin{Bmatrix} x(0) \\ \dot{x}(0) \end{Bmatrix} = \begin{Bmatrix} x_o \\ \dot{x}_0 \end{Bmatrix} \tag{5.19}$$

Let $t_{k+1} = t\,,\ t_k = t_0\,,\ \Delta t = t - t_0$

Hence,

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}} \int_{t_k}^{t_{k+1}} \boldsymbol{e}^{-\boldsymbol{A}s}\,\boldsymbol{F}(s)ds \tag{5.20}$$

The objective of the numerical analysis using the integration method is to integrate the forcing function as given in Equation (5.20). Since the forcing function is usually given in digitized for, approximation of this forcing function within the time interval is necessary. Two methods are used to integrate the forcing function (Hart and Wong, 2000).

(i) Delta forcing function method
(ii) Constant forcing function method

In delta forcing function method, the forcing function is digitized using a series of delta functions. The forcing function is represented by,

$$\boldsymbol{F}(s) = \boldsymbol{F}_k\, \delta(s - t_k)\Delta t = \begin{Bmatrix} 0 \\ F_k/m \end{Bmatrix} \delta(s - t_k)\Delta t \quad ; \; t_k \le s \le t_{k+1} \tag{5.21}$$

Substituting Equation (5.21) into Equation (5.20),

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}} \int_{t_k}^{t_{k+1}} \boldsymbol{e}^{-\boldsymbol{A}s}\, \boldsymbol{F}_k \delta(s - t_k)\Delta t ds \tag{5.22}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}} \left[\int_{t_k}^{t_{k+1}} \boldsymbol{e}^{-\boldsymbol{A}s}\, \delta(s - t_k) ds \right] \boldsymbol{F}_k \Delta t \tag{5.23}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}}\, \boldsymbol{e}^{-\boldsymbol{A}t_k} \boldsymbol{F}_k \Delta t \tag{5.24}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{z}_{\mathrm{k}} + \Delta t \boldsymbol{e}^{\boldsymbol{A}\Delta t} \boldsymbol{F}_k \tag{5.25}$$

In constant forcing function method, the forcing function is assumed to be constant within the time interval. The value of the force in the interval is equal to the values of the force at the beginning of the interval.

Therefore,

$$\boldsymbol{F}(s) = \boldsymbol{F}_k = \begin{Bmatrix} 0 \\ F_k/m \end{Bmatrix} \quad ; \; t_k \le s \le t_{k+1} \tag{5.26}$$

Substituting Equation (5.26) into Equation (5.20),

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}} \int_{t_k}^{t_{k+1}} \boldsymbol{e}^{-\boldsymbol{A}s}\, \boldsymbol{F}_k ds \tag{5.27}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{e}^{\boldsymbol{A}t_{k+1}}\, \boldsymbol{A}^{-1} (\boldsymbol{e}^{-\boldsymbol{A}t_k} - \boldsymbol{e}^{-\boldsymbol{A}t_{k+1}}) \boldsymbol{F}_k \tag{5.28}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{A}^{-1} (\boldsymbol{e}^{\boldsymbol{A}\Delta t} - \boldsymbol{I}) \boldsymbol{F}_k \tag{5.29}$$

Now, considering the earthquake ground excitation to SDOF system, the forcing function is given by,

$$\boldsymbol{F}_k = -m\, a_k \tag{5.30}$$

where, a_k is the ground acceleration at time step k.

The external force vector,

$$\boldsymbol{F}_k = \begin{Bmatrix} 0 \\ F_k/m \end{Bmatrix} = \begin{Bmatrix} 0 \\ -m(a_k/m) \end{Bmatrix} = \begin{Bmatrix} 0 \\ -a_k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} a_k \tag{5.31}$$

Substituting Equation (5.31) into Equation (5.29),

$$\boldsymbol{z}_{k+1} = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{A}^{-1}\,(\boldsymbol{e}^{\boldsymbol{A}\Delta t} - \boldsymbol{I})\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} a_k \tag{5.32}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{A_d}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E_d}\boldsymbol{a}_{\mathrm{k}} \tag{5.33}$$

where,

$$\boldsymbol{A_d} = \boldsymbol{e}^{\boldsymbol{A}\Delta t} \tag{5.34}$$

$$\boldsymbol{E_d} = \boldsymbol{A}^{-1}\,(\boldsymbol{e}^{\boldsymbol{A}\Delta t} - \boldsymbol{I})\boldsymbol{E} \tag{5.35}$$

$$\boldsymbol{E} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} \tag{5.36}$$

$$\dot{\boldsymbol{z}}_{k+1} = \boldsymbol{A}\boldsymbol{z}_{\mathrm{k+1}} + \boldsymbol{E}\boldsymbol{a}_{\mathrm{k+1}} \tag{5.37}$$

where,

$$\boldsymbol{z}_{k+1} = \begin{Bmatrix} x_{k+1} \\ \dot{x}_{k+1} \end{Bmatrix} \quad \text{and} \quad \dot{\boldsymbol{z}}_{k+1} = \begin{Bmatrix} \dot{x}_{k+1} \\ \ddot{x}_{k+1} \end{Bmatrix} \tag{5.38}$$

Equations (5.33) and (5.37) will give the solution of equation of motion in terms of the response quantities, displacements, velocity and acceleration (Hart and Wong, 2000).

5.4 Effects of Support Excitations

It is very important to perform dynamic analysis for the structures subjected to earthquake induced ground motions. The support induced vibrations cause deformations and stresses in the structural systems. The support excitations can be divided into two types:

(i) Single-support excitation
(ii) Multi-support excitation

In single-support excitation, it is assumed that all the supports undergo an identical (uniform) ground motion. In other words, due to the same ground motion at all supports, the supports move as one rigid base as shown in Figure 5.2. Hence, the masses attached to dynamics degrees of freedom are excited by the ground motion. For example, tall buildings, towers, chimneys etc. for which the distances between the supports are not very large compared with the predominant wave length of the ground motion (Chopra, 2007; Datta, 2010)

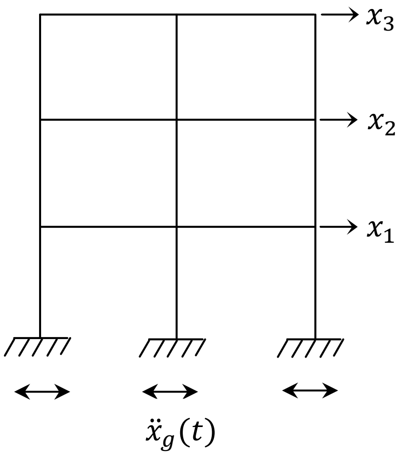


Figure 5.2 A system subjected to single-support excitation

In multi-support excitations, the ground/support motions or excitations are different at difference supports as shown in Figure 5.3. For the same travelling wave of an earthquake, the time histories of ground motion at two supports could be different if the two supports are separated by a large distance. This is the case because the travel time of the wave between any two supports is not sufficiently negligible to make the assumption that the ground motions are the same at the two supports. For examples, big network of pipe lines, very long tunnels, long dams, bridges etc. Although the piping may not be especially long, its ends are connected to different locations of the main structure and would therefore experience different motions during as earthquake (Chopra, 2007; Datta, 2010).

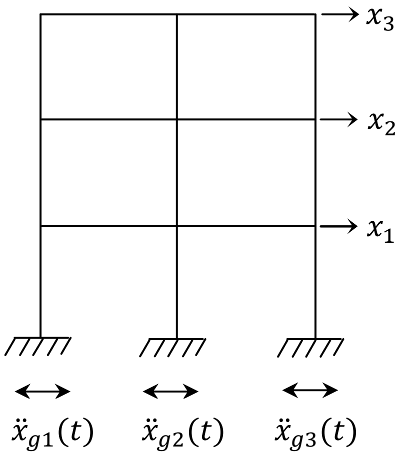


Figure 5.3 A system subjected to multi-support excitations

5.5 Equations of Motion for MDOF System with Single-Support Excitation

For single-support excitation, the same earthquake ground motion excites all the masses. As discussed in Section 5.2, the Equation (5.7) is the equation of motion for SDOF system, which can be extended for the multi degree of freedoms system (MDOF) as follows:

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}}(t) + \boldsymbol{C}\,\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\,\boldsymbol{x}(t) = -\,\boldsymbol{M}\,\boldsymbol{\Gamma}\,\ddot{\boldsymbol{x}}_{\boldsymbol{g}}(t) \tag{5.39}$$

where,
n is the number of degrees of freedom
r is the number of components of input ground motion
$\boldsymbol{M}$ is the mass matrix of the system of size n x n
$\boldsymbol{K}$ is the stiffness matrix of the system of size n x n
$\boldsymbol{C}$ is the damping matrix of the system of size n x n
$\boldsymbol{x}$ is the relative displacement vector of size n x 1
$\dot{\boldsymbol{x}}$ is the relative velocity vector of size n x 1
$\ddot{\boldsymbol{x}}$ is the absolute acceleration vector of size n x 1
$\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$ is the ground acceleration vector of size r x 1
$\boldsymbol{\Gamma}$ is the influence coefficient matrix of size n x r

For example, for the single component of earthquake ground motion, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}} = \ddot{x}_g$

for the two component of earthquake ground motion, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}} = \begin{Bmatrix} \ddot{x}_{g1} \\ \ddot{x}_{g2} \end{Bmatrix}$

and, for the three component of earthquake ground motion, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}} = \begin{Bmatrix} \ddot{x}_{g1} \\ \ddot{x}_{g2} \\ \ddot{x}_{g3} \end{Bmatrix}$

$\boldsymbol{\Gamma}$ is the influence coefficient matrix of size n x r, having '1' for elements corresponding to degree of freedom in the direction of the applied ground motion and '0' for other degree of freedom

For example, for two storey lumped mass system and hence 2 degrees of freedom system with single component of ground motion, $\boldsymbol{\Gamma} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

with two component of ground motion, $\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and, with three component of ground motion, $\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5.5.1 Equations of Motion in State Space for MDOF System with Single-Support Excitation and its Solution

Equation (5.7) represents the equation of motion for SDOF system and it can be expressed in form of state space as shown in Equation (5.16). In a similar way, the Equation (5.39) represents the equations of motion for MDOF system, for which the state space expression can be extended as follows (Hart and Wong, 2000).

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.40}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\,\boldsymbol{z} + \boldsymbol{f} \tag{5.41}$$

where,

$$\dot{\boldsymbol{z}} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix};\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix};\ \boldsymbol{z} = \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix};\ \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.42}$$

The above Equation (5.42) is the state space form of Equation (5.39) in terms of relative motion of the mass.

Further, the equations of motion as defined in Equation (5.39) can be further extended in terms of absolute (total) motion of the mass and it can be written as,

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}}^t + \boldsymbol{C}\,\dot{\boldsymbol{x}}^t + \boldsymbol{K}\,\boldsymbol{x}^t = \boldsymbol{C}\,\dot{\boldsymbol{x}}_{\boldsymbol{g}} + \boldsymbol{K}\,\boldsymbol{x}_{\boldsymbol{g}} \tag{5.43}$$

where,

$$\begin{aligned}&\text{Absolute (total) displacement, } \boldsymbol{x^t} = \boldsymbol{x} + \boldsymbol{x_g}\\&\text{Absolute (total) velocity, } \dot{\boldsymbol{x}}^{\boldsymbol{t}} = \dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}_{\boldsymbol{g}}\\&\text{Absolute (total) acceleration, } \ddot{\boldsymbol{x}}^{\boldsymbol{t}} = \ddot{\boldsymbol{x}} + \ddot{\boldsymbol{x}}_{\boldsymbol{g}}\end{aligned} \tag{5.44}$$

Rewriting Equation (5.43) in state space form as discussed earlier (Hart and Wong, 2000),

$$\dot{\boldsymbol{z}}^{\boldsymbol{t}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x^t} \\ \dot{\boldsymbol{x}}^{\boldsymbol{t}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{K} & \boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x_g} \\ \dot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.45}$$

$$\dot{\boldsymbol{z}}^{\boldsymbol{t}} = \boldsymbol{A}\,\boldsymbol{z^t} + \boldsymbol{F}\,\boldsymbol{f} \tag{5.46}$$

where,

$$\begin{gathered}\dot{\boldsymbol{z}}^{\boldsymbol{t}} = \begin{Bmatrix} \dot{\boldsymbol{x}}^{\boldsymbol{t}} \\ \ddot{\boldsymbol{x}}^{\boldsymbol{t}} \end{Bmatrix};\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix};\ \boldsymbol{z^t} = \begin{Bmatrix} \boldsymbol{x^t} \\ \dot{\boldsymbol{x}}^{\boldsymbol{t}} \end{Bmatrix};\\ \boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{K} & \boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix};\ \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{x_g} \\ \dot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix}\end{gathered} \tag{5.47}$$

The above Equation (5.46) is the state space form of Equation (5.43) in terms of absolute (total) motion of the mass. The solution of above derived equation of motion can be obtained by using the procedure as defined in Section 5.3 using the Equations (5.33) and (5.37).

5.6 Equations of Motion for MDOF System with Multi-Support Excitations

In the case when a linear elastic structure is supported at more than one support and is subjected to different input ground motions, the formulation of the response to each input component is different from a system having uniform support excitation. The difference is that when the multiple supports move independently of each other, they induce quasi-static stresses that must be considered in addition to the dynamics response effects resulting from inertial forces. The frame as shown in Figure 5.4 represents the various degrees of freedoms (Chopra, 2007; Datta, 2010).

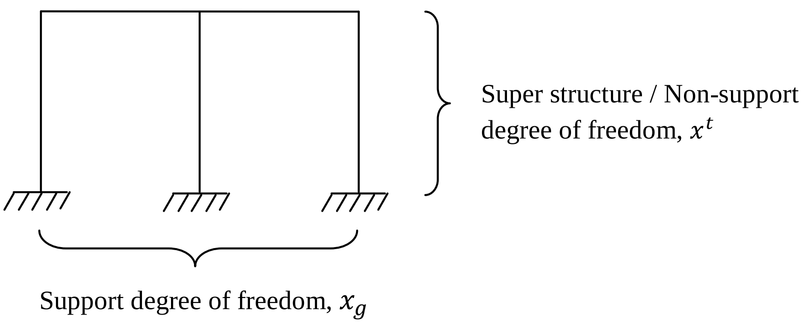


Figure 5.4 A frame representing the degrees of freedom

For the system with single-support excitation, the total displacement of the super structure is obtained by adding the input ground motion to the relative displacements of the structure with respect to the supports. This relationship is given by,

$$x^t(t) = x(t) + \{1\}x_g(t) \tag{5.48}$$

where, the vector {1} expresses the fact that a unit static translation of the base of the structure produces directly a unit displacement of all degrees of freedom.

For the system with multi-support excitations, where the relative displacements are not measured parallel to the ground motion, the support motions at any instant of time are different for the various supports and therefore, the total displacements of the super structure / non-support degrees of freedom may be expressed as the sum of the relative displacements of the structure with respect to the supports and the quasi-static displacements (x_s) that would result from a static-support displacement (or the displacements produced at non-support degrees of freedom due to quasi-static motions of the supports) (Chopra, 2007; Clough and Penzien, 1993; Datta, 2010).

$$\boldsymbol{x^t}(t) = \boldsymbol{x}(t) + \boldsymbol{x_s}(t) \tag{5.49}$$

The quasi-static displacements can be expressed conveniently by an influence coefficient vector '$\boldsymbol{\Gamma}$' which represents the displacements resulting from the unit support displacements.

Thus,

$$\boldsymbol{x_s} = \boldsymbol{\Gamma}\, \boldsymbol{x_g} \tag{5.50}$$

and

$$\boldsymbol{x^t} = \boldsymbol{x} + \boldsymbol{\Gamma}\, \boldsymbol{x_g} \tag{5.51}$$

where, $\boldsymbol{\Gamma}$ is an influence coefficient matrix of size n x r
In which n is the number of super structure non-supports degree of freedom and r is the number of components of input ground motion

The equations of motion for MDOF system with multi-support excitations can be written as follows (Chopra, 2007; Datta, 2010),

$$\begin{bmatrix} \boldsymbol{M}_{ss} & \boldsymbol{M}_{sg} \\ \boldsymbol{M}_{gs} & \boldsymbol{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}^{\boldsymbol{t}} \\ \ddot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{C}_{ss} & \boldsymbol{C}_{sg} \\ \boldsymbol{C}_{gs} & \boldsymbol{C}_{gg} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}}^{\boldsymbol{t}} \\ \dot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{K}_{sg} \\ \boldsymbol{K}_{gs} & \boldsymbol{K}_{gg} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}^{\boldsymbol{t}} \\ \boldsymbol{x}_{\boldsymbol{g}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{P}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.52}$$

where,

$\boldsymbol{M}_{ss}$ is the mass matrix corresponding to super structure / non-support degrees of freedom
$\boldsymbol{M}_{gg}$ is the mass matrix corresponding to support degrees of freedom
$\boldsymbol{M}_{sg}$ and $\boldsymbol{M}_{gs}$ are the coupling mass matrices that expresses the inertia forces in super structure degrees of freedom of the structure due to motions of the supports
The terms of damping and stiffness matrices are defined in similar ways
$\boldsymbol{x^t}$ is the vector of total displacements corresponding to super structure degrees of freedom
$\boldsymbol{x_g}$ is the vector of input ground displacements at the supports
$\dot{\boldsymbol{x}}^{\boldsymbol{t}}$, $\dot{\boldsymbol{x}}_{\boldsymbol{g}}$, $\ddot{\boldsymbol{x}}^{\boldsymbol{t}}$, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$ are the velocity and accelerations vectors defined in similar ways
$\boldsymbol{P_g}$ is the vector of forces generated at the support degrees of freedom.

In Equation (5.52), no external forces are applied along the super structure degrees of freedom and the matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ can be determined from the properties of structure. Further, to write the governing equations in a form similar to the formulation for single support excitation as per Equation (5.51) and hence separating the displacements into two parts,

$$\begin{Bmatrix} \boldsymbol{x}^{t} \\ \boldsymbol{x}_{g} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{x}_{s} \\ \boldsymbol{x}_{g} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{Bmatrix} \tag{5.53}$$

In the above equation, vector $\boldsymbol{x}_{s}$ is the vector of structural displacements, due to static application of the prescribed support displacements, $\boldsymbol{x}_{g}$ at each time instant. To find the quasi-static displacements, $\boldsymbol{x}_{s}$, produced due to the support displacements, $\boldsymbol{x}_{g}$,the quasi-static equation of equilibrium can be written as (Chopra, 2007; Datta, 2010),

$$\begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{K}_{sg} \\ \boldsymbol{K}_{gs} & \boldsymbol{K}_{gg} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_{s} \\ \boldsymbol{x}_{g} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{P}_{g}{}^{s} \end{Bmatrix} \tag{5.54}$$

where, $\boldsymbol{P}_{g}{}^{s}$ are the support forces necessary to statically impose displacements, $\boldsymbol{x}_{g}$, that vary with time. Further, $\boldsymbol{P}_{g}{}^{s} = \boldsymbol{0}$, if the structure is statically determinate or if the support undergoes rigid body motion.

From Equation (5.54),

$$\boldsymbol{K}_{ss}\, \boldsymbol{x}_{s} + \boldsymbol{K}_{sg}\, \boldsymbol{x}_{g} = \boldsymbol{0} \tag{5.55}$$

Simplifying the above equation gives,

$$\boldsymbol{x}_{s} = \boldsymbol{\Gamma}\, \boldsymbol{x}_{g} \tag{5.56}$$

where,

$$\boldsymbol{\Gamma} = -\boldsymbol{K}_{ss}{}^{-1} \boldsymbol{K}_{sg} \tag{5.57}$$

Equation (5.56) is showing the quasi-static displacements, $\boldsymbol{x}_{s}$, in terms of the specified support displacements, $\boldsymbol{x}_{g}$. Further, substituting Equation (5.56) into (5.55),

$$(\boldsymbol{K}_{ss}\, \boldsymbol{\Gamma} + \boldsymbol{K}_{sg})\, \boldsymbol{x}_{g} = \boldsymbol{0} \tag{5.58}$$

Now, to calculate the response of non-support degrees of motion the following equation can be written from Equation (5.52),

$$\boldsymbol{M}_{ss}\, \ddot{\boldsymbol{x}}^{t} + \boldsymbol{M}_{sg}\, \ddot{\boldsymbol{x}}_{g} + \boldsymbol{C}_{ss}\, \dot{\boldsymbol{x}}^{t} + \boldsymbol{C}_{sg}\, \dot{\boldsymbol{x}}_{g} + \boldsymbol{K}_{ss}\, \boldsymbol{x}^{t} + \boldsymbol{K}_{sg}\, \boldsymbol{x}_{g} = \boldsymbol{0} \tag{5.59}$$

Hence,

$$\boldsymbol{M}_{ss}\, \ddot{\boldsymbol{x}}^{t} + \boldsymbol{C}_{ss}\, \dot{\boldsymbol{x}}^{t} + \boldsymbol{K}_{ss}\, \boldsymbol{x}^{t} = -\boldsymbol{M}_{sg}\, \ddot{\boldsymbol{x}}_{g} - \boldsymbol{C}_{sg}\, \dot{\boldsymbol{x}}_{g} - \boldsymbol{K}_{sg}\, \boldsymbol{x}_{g} \tag{5.60}$$

In most cases, there are few non-zero terms in the mass coupling matrix and damping matrix, and when present they are generally relatively small and hence those two terms usually contributes little and hence can be ignored.

Therefore,

$$\boldsymbol{M}_{ss}\, \ddot{\boldsymbol{x}}^{t} + \boldsymbol{C}_{ss}\, \dot{\boldsymbol{x}}^{t} + \boldsymbol{K}_{ss}\, \boldsymbol{x}^{t} = -\boldsymbol{K}_{sg}\, \boldsymbol{x}_{g} \tag{5.61}$$

Now, substituting Equation (5.51) and its similar velocity and acceleration components into Equation (5.60),

$$\boldsymbol{M}_{ss}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}_{ss}\,\dot{\boldsymbol{x}} + \boldsymbol{K}_{ss}\,\boldsymbol{x} \;= -\left(\boldsymbol{M}_{sg} + \Gamma\boldsymbol{M}_{ss}\right)\ddot{\boldsymbol{x}}_{\boldsymbol{g}} - \left(\boldsymbol{C}_{sg} + \Gamma\boldsymbol{C}_{ss}\right)\dot{\boldsymbol{x}}_{\boldsymbol{g}} - \left(\boldsymbol{K}_{sg} + \;\Gamma\boldsymbol{K}_{ss}\right)\boldsymbol{x}_{\boldsymbol{g}} \quad (5.62)$$

As derived earlier in Equation (5.58), $(\boldsymbol{K}_{ss}\,\Gamma\; + \;\boldsymbol{K}_{sg})\,\boldsymbol{x}_{\boldsymbol{g}} = \boldsymbol{0}$ and the term $\boldsymbol{M}_{sg}$ denoting the inertia coupling which can be neglected for most structures. Another assumption for neglecting $\boldsymbol{M}_{sg}$ is that for structures with mass idealized as lumped at the degree of freedom, the mass matrix is diagonal, implying that $\boldsymbol{M}_{sg}$ is null matrix and $\boldsymbol{M}_{ss}$ is diagonal. Also, the contribution of the damping term $\left(\boldsymbol{C}_{sg} + \Gamma\boldsymbol{C}_{ss}\right)\dot{\boldsymbol{x}}_{\boldsymbol{g}}$ is very small and can be neglected.

Hence, $$\boldsymbol{M}_{ss}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}_{ss}\,\dot{\boldsymbol{x}} + \boldsymbol{K}_{ss}\,\boldsymbol{x} \;= -\boldsymbol{M}_{ss}\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \quad (5.63)$$

The above Equation (5.63) is the equations of motion for the MDOF system subjected to multi-support excitation and is similar in a form with Equation (5.44) of SDOF system subjected to single-support excitation. The matrix $\boldsymbol{\Gamma}$ for a single support excitation is obtained straight away whereas, for multi-support excitations it is obtained from a static analysis of structure for relative movements.

5.6.1 Equations of Motion in State Space for MDOF System with Multi-Support Excitations and its Solution

Equation (5.63) represents the equations of motion for MDOF system subjected to multi-support excitations, for which the state space expression can be expressed as follows (Hart and Wong, 2000).

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{K}_{ss} & -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{C}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} \quad (5.64)$$

$$\dot{\boldsymbol{z}} = \;\boldsymbol{A}\,\boldsymbol{z} + \;\boldsymbol{f} \quad (5.65)$$

where,

$$\dot{\boldsymbol{z}} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix};\; \boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{K}_{ss} & -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{C}_{ss} \end{bmatrix};\; \boldsymbol{z} = \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix};\boldsymbol{f} = \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \end{Bmatrix} \quad (5.66)$$

The above Equation (5.65) is the state space form of Equation (5.63) in terms of relative motion of the mass. The solution of above derived equation of motion can be obtained by using the procedure as defined in Section 5.3 using the Equations (5.33) and (5.37).

Further, the equations of motion as defined in Equation (5.63) can be further extended in terms of absolute (total) motion of the mass and it can be written as,

$$\boldsymbol{M}_{ss}\,\ddot{\boldsymbol{x}}^t + \;\boldsymbol{C}_{ss}\dot{\boldsymbol{x}}^t + \boldsymbol{K}_{ss}\,\boldsymbol{x}^t \;= -\,\boldsymbol{K}_{sg}\,\boldsymbol{x}_{\boldsymbol{g}} \quad (5.67)$$

where,

Absolute (total) displacement, $\boldsymbol{x}^{\boldsymbol{t}} = \boldsymbol{x} + \;\boldsymbol{x}_{\boldsymbol{g}}$

Absolute (total) velocity, $\dot{\boldsymbol{x}}^{\boldsymbol{t}} = \dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}_{\boldsymbol{g}}$ (5.68)
Absolute (total) acceleration, $\ddot{\boldsymbol{x}}^{\boldsymbol{t}} = \ddot{\boldsymbol{x}} + \ddot{\boldsymbol{x}}_{\boldsymbol{g}}$

Rewriting Equation (5.67) in state space form as discussed earlier (Hart and Wong, 2000),

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{K}_{ss} & -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{C}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{K}_{sg}\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{x}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.69}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\,\boldsymbol{z} + \bar{\boldsymbol{f}} \tag{5.70}$$

where,

$$\dot{\boldsymbol{z}} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{Bmatrix};\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{K}_{ss} & -\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{C}_{ss} \end{bmatrix};\ \boldsymbol{z} = \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix};\ \bar{\boldsymbol{f}} = \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{K}_{sg}\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{x}_{\boldsymbol{g}} \end{Bmatrix} \tag{5.71}$$

The above Equation (5.71) is the state space form of Equation (5.67) in terms of absolute (total) motion of the mass.

5.7 MATLAB Steps for Computing the Response

Step 1:
Generate the mass matrix by modeling the system as the lumped mass model or continuous system model. Mass matrix will be $\boldsymbol{M}(= \boldsymbol{M}_{ss})$ of size n x n. where 'n' is the number of super structure / non-support degrees of freedom.

Step 2:
Generate the overall stiffness matrix $\boldsymbol{K}_{\boldsymbol{T}}$ using the static analysis procedure by calculating the stiffness influence coefficients. As discussed earlier in Section 5.6, the general form of stiffness matrix will be as follows,

$$\boldsymbol{K}_{\boldsymbol{T}} = \left[\begin{array}{c|c} \boldsymbol{K}_{ss} & \boldsymbol{K}_{sg} \\ \hline \boldsymbol{K}_{gs} & \boldsymbol{K}_{gg} \end{array}\right] \tag{5.72}$$

Recall again, 'n' is the number of super structure / non-supports degrees of freedom, whereas 'r' is the number of components of input ground motion (or number of support degrees of freedom). Hence, size of $\boldsymbol{K}_{ss}$ will be 'n x n', size of $\boldsymbol{K}_{sg}$ will be 'n x r', size of $\boldsymbol{K}_{gs}$ will be 'r x n', size of $\boldsymbol{K}_{gg}$ will be 'r x r' and the overall size of $\boldsymbol{K}_{\boldsymbol{T}}$ will be '(n + r) x (n + r)'.

Step 3:
Calculate the eigen values and natural frequencies (ω_i).

Step 4:
Generate the Rayleigh's damping matrix, $\boldsymbol{C}$ by assuming percentage of critical damping (ξ) for all modes by using following equation.

$$\boldsymbol{C} = a_o\,\boldsymbol{M} + a_1\,\boldsymbol{K} \tag{5.73}$$

$$a_o = \frac{2\xi\omega_1\omega_2}{\omega_1 + \omega_2} \text{ and } a_1 = \frac{2\xi}{\omega_1 + \omega_2} \tag{5.74}$$

Step 5:

Derive the influence coefficient matrix, $\boldsymbol{\Gamma}$.

For single support excitation, it may be obtained by arranging '1' and '0' at proper places corresponding to degrees of freedom as discussed in Section 5.5.

For multi support excitations, it may be calculated by using Equation (5.47) as derived in Section 5.6. It is rewritten as follows,

$$\boldsymbol{\Gamma} = -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sg} \tag{5.75}$$

Step 6:

Generate the ground motion (generally, acceleration) vector, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$ corresponding to the support degrees of freedom considering the effects of time delay and the size of $\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$ will be 'r x 1', where 'r' is number of support degrees of freedom.

Step 7:

Calculate the state transition matrix, $\boldsymbol{A_d}$ as discussed in previous sections.

$$\boldsymbol{A_d} = \boldsymbol{e}^{\boldsymbol{A}\Delta t} \tag{5.76}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_{ss}^{-1}\boldsymbol{K}_{ss} & -\boldsymbol{M}_{ss}^{-1}\boldsymbol{C}_{ss} \end{bmatrix} \tag{5.77}$$

and 'Δt' is the time step considered corresponding to input ground motion.
For single support excitation, $\boldsymbol{M}_{ss} = \boldsymbol{M}$, $\boldsymbol{C}_{ss} = \boldsymbol{C}$, $\boldsymbol{K}_{ss} = \boldsymbol{K}$ that means without considering the coupling effects.

Step 8:

Calculate the state vector, $\boldsymbol{z}$ for each time step, Δt as follows,

(i) For Single-Support Excitation:

$$\boldsymbol{z}_{k+1} = \boldsymbol{A_d}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E_d}\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathrm{k}}} \tag{5.78}$$

where,

$$\boldsymbol{E_d} = \boldsymbol{A}^{-1}(\boldsymbol{A_d} - \boldsymbol{I})\boldsymbol{E} \tag{5.79}$$

and

$$\boldsymbol{E} = \begin{Bmatrix} \boldsymbol{0}_{\mathbf{nx1}} \\ -\boldsymbol{1}_{\mathbf{nx1}} \end{Bmatrix} \tag{5.80}$$

(ii) For Multi-Support Excitations:

$$\boldsymbol{z}_{k+1} = \boldsymbol{A_d}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E_d} \tag{5.81}$$

where, $$\boldsymbol{E} = \begin{Bmatrix} \mathbf{0}_{\mathbf{nx1}} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathbf{nx1}}} \end{Bmatrix} \tag{5.82}$$

The solution of above equation gives the responses of relative displacement and relative velocity at super structure degrees of freedom as follows,

$$\boldsymbol{z} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} \tag{5.83}$$

<u>Step 9:</u>

Calculate the state vector, $\dot{\boldsymbol{z}}$ for each time step, Δt as follows,

(i) For Single-Support Excitation:

$$\dot{\boldsymbol{z}}_{\boldsymbol{k+1}} = \boldsymbol{A}\,\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E}\,\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathrm{k+1}}} \tag{5.84}$$

where, $\boldsymbol{E}$ is as defined in Step 8 (i).

(ii) For Multi-Support Excitations:

$$\dot{\boldsymbol{z}}_{\boldsymbol{k+1}} = \boldsymbol{A}\,\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E} \tag{5.85}$$

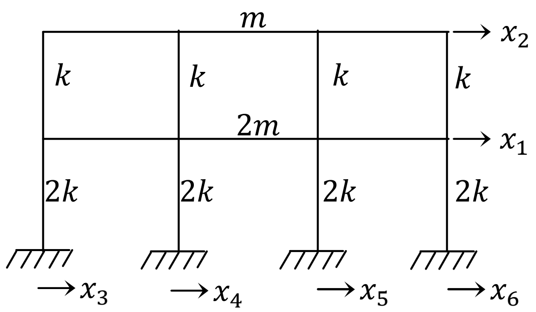
where, $\boldsymbol{E}$ is as defined in Step 8 (ii).

The solution of above equation gives the responses of relative velocity and relative acceleration at super structure degrees of freedom as follows,

$$\dot{\boldsymbol{z}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ . \\ . \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} \tag{5.86}$$

Example 5.1 For the multi-bay portal frame as shown in Figure 5.5a, calculate the displacements x_1 and x_2 when subjected to El-Centro, 1940 (N-S component) earthquake ground motion for the following cases.
Case (i) Considering the same excitation at all supports (uniform excitation)
Case (ii) Considering multi-support excitations with a time delay of 5 s between supports
Assume percentage of critical damping as 5 %, k/m = 100 and all members are inextensible and EI values same for all members.

Solution:



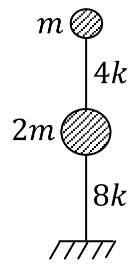


Figure 5.5 (a) A multi-bay portal frame (b) Lumped mass model of frame

Calculation of General Elements :

Step 1: Generation of mass matrix

With the help of lumped mass assumption, the frame can be represented as shown in Figure 5.5b and the mass matrix can be expressed as follows,

$$\boldsymbol{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

Step 2: Generation of stiffness matrix

The overall stiffness matrix for the given system is to be calculated by considering the effects of coupling between super structure and support degrees of freedom. The stiffness influence coefficients, k_{ij} are derived to assemble the stiffness matrix. Where k_{ij} is the force required along degree of freedom, i due to unit displacement at degree of freedom, j.

(a) Imposing unit displacement at degree of freedom, 1, i.e. x_1 = 1

To obtain the first column of the stiffness matrix, imposing x_1 = 1 and zero displacement at all other degrees of freedom. The forces necessary at the top and bottom of each storey corresponding to all degrees of freedom to maintain the deflected shape as shown in Figure 5.6a are expressed in terms of storey stiffnesses.

k_{11} = 4(2k)+4(k) = 12 k

k_{21} = - 4(k) = - 4 k

k_{31} = k_{41} = k_{51} = k_{61} = - 2(k) = -2 k

(b) Imposing unit displacement at degree of freedom, 2, i.e. x_2 = 1

Similarly, to obtain the second column of the stiffness matrix, imposing x_2 = 1 and zero displacement at remaining degrees of freedom. The forces necessary at the top and bottom of each storey corresponding to all degrees of freedom to maintain the deflected shape as shown in Figure 5.6b are expressed as follows,

k_{12} = - 4(k) = - 4 k

k_{22} = 4(k) = 4 k

k_{32} = k_{42} = k_{52} = k_{62} = 0

(c) Imposing unit displacement at degree of freedom, 3, i.e. x_3 = 1

In the above similar manner, imposing x_3 = 1 and zero displacement at remaining degrees of freedom. The forces as shown in Figure 5.6c are expressed as follows,
k_{13} = - 2 k
k_{23} = 0
k_{33} = 2 k
k_{43} = k_{53} = k_{63} = 0

(d) Imposing unit displacement at degree of freedom, 4, i.e. x_4 = 1
Similarly imposing x_4 = 1 and zero displacement at remaining degrees of freedom. The forces as shown in Figure 5.6d are expressed as follows,
k_{14} = - 2 k
k_{24} = k_{34} = 0
k_{44} = 2 k
k_{54} = k_{64} = 0

(e) Imposing unit displacement at degree of freedom, 5, i.e. x_5 = 1

Now, imposing x_5 = 1 and zero displacement at remaining degrees of freedom. The forces as shown in Figure 5.6e are expressed as follows,
k_{15} = - 2 k
k_{25} = k_{35} = k_{45} = 0
k_{55} = 2 k
k_{65} = 0

(f) Imposing unit displacement at degree of freedom, 6, i.e. x_6 = 1

Finally, imposing x_6 = 1 and zero displacement at remaining degrees of freedom. The forces as shown in Figure 5.6f are expressed as follows,
k_{16} = - 2 k
k_{26} = k_{36} = k_{46} = k_{56} = 0
k_{66} = 2 k

Here, for the considered frame, n = 2 and r = 4. Hence assembling the above derived stiffness influence coefficients to get the overall stiffness matrix, $\boldsymbol{K_T}$.

Therefore, $$\boldsymbol{K_T} == \left[\begin{array}{c|c} \boldsymbol{K}_{ss} & \boldsymbol{K}_{sg} \\ \hline \boldsymbol{K}_{gs} & \boldsymbol{K}_{gg} \end{array}\right] = \left[\begin{array}{cc|cccc} 12k & -4k & -2k & -2k & -2k & -2k \\ -4k & 4k & 0 & 0 & 0 & 0 \\ \hline -2k & 0 & 2k & 0 & 0 & 0 \\ -2k & 0 & 0 & 2k & 0 & 0 \\ -2k & 0 & 0 & 0 & 2k & 0 \\ -2k & 0 & 0 & 0 & 0 & 2k \end{array}\right]$$

<u>Step 3: Generation of damping matrix</u>

Assuming the Rayleigh's mass and stiffness proportional damping and for that considering critical damping, ξ = 5 % in all modes. As per the given data consider, k = 1000 *N/m* and m = 10 *kg* and from the eigen values analysis, the eigen values and hence natural frequencies are obtained as follows,

ω_1 = 14.1421 rad/sec and ω_2 = 28.2843 rad/sec.

Further, the constants to derive the damping matrix can be obtained as follows,

$a_o = \frac{2\xi\omega_1\omega_2}{\omega_1 + \omega_2}$ = 0.9428 and $a_1 = \frac{2\xi}{\omega_1 + \omega_2}$ = 0.002357

Considering $\boldsymbol{K} = \boldsymbol{K}_{ss}$ and using the Equation (5.73), the damping matrix can be derived as follows,

$$\boldsymbol{C} = = a_o\,\boldsymbol{M} + a_1\,\boldsymbol{K} = \begin{bmatrix} 47.1405 & -9.4281 \\ -9.4281 & 18.8562 \end{bmatrix}$$

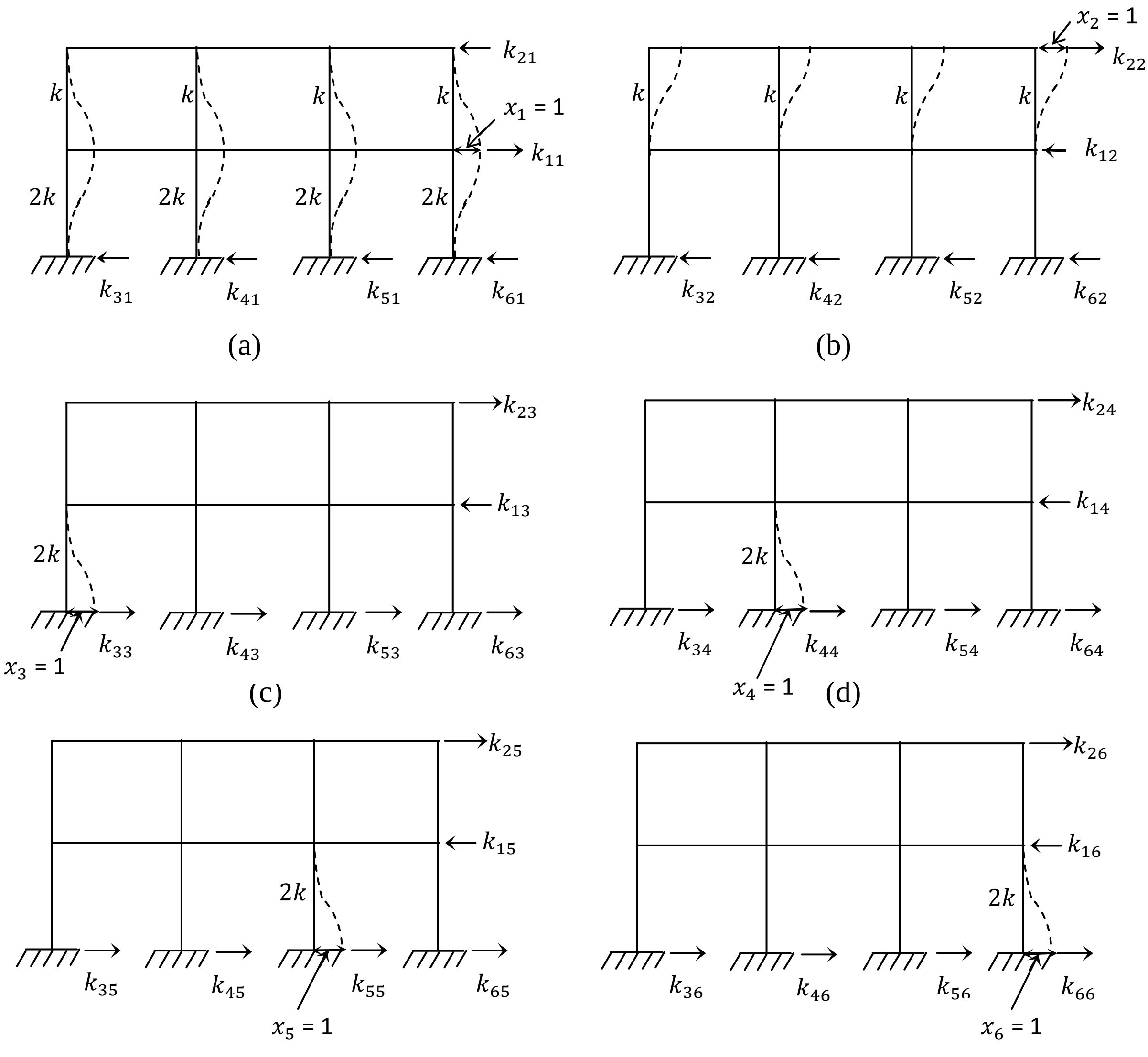


Figure 5.6 A Frame with stiffness influence coefficients

Case (i) Considering the single support (uniform) excitation at all supports

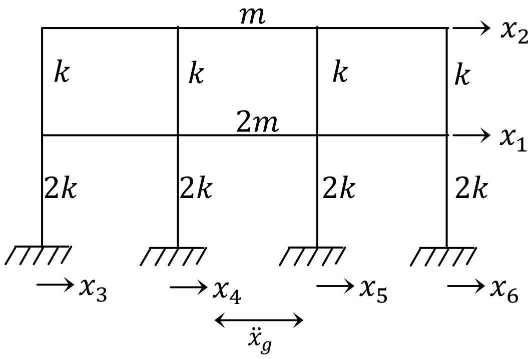


Figure 5.7 A Frame subjected to single support excitation

For the system as shown in above figure, the equations of motion can be written as follows,

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = -\,\boldsymbol{M}\,\boldsymbol{I}\,\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + [a_o\,\boldsymbol{M} + a_1\,\boldsymbol{K}] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 12k & -4k \\ -4k & 4k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$
$$= -\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{x}_g$$

Now, the system is subjected to uniform El-Centro earthquake ground motion as shown in Figure 5.8 (http://www.vibrationdata.com/elcentro.dat) , hence $\ddot{x}_g$ will be having size of '1x1'. The time step, Δt is considered as 0.02 s. Assume k = 1000 *N/m* and m = 10 *kg*. Further, with the state space method using the constant forcing function method, the solution of equations of motion can be obtained as follows,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -600 & 200 & -2.3570 & 0.4714 \\ 400 & -400 & 0.9428 & -1.8856 \end{bmatrix}$$

$$\boldsymbol{A_d} = \boldsymbol{e}^{\boldsymbol{A}\Delta t} = \begin{bmatrix} 0.8849 & 0.0378 & 0.0188 & 0.0003 \\ 0.0757 & 0.9228 & 0.0007 & 0.0191 \\ -11.1235 & 3.6160 & 0.8410 & 0.0460 \\ 7.2319 & -7.5075 & 0.0921 & 0.8871 \end{bmatrix}$$

$$\boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\boldsymbol{E_d} = \boldsymbol{A}^{-1}\,(\boldsymbol{A_d} - \boldsymbol{I})\boldsymbol{E} = \begin{bmatrix} -0.0001949 \\ -0.0001987 \\ -0.0191134 \\ -0.0198025 \end{bmatrix}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{A_d}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E_d}\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathrm{k}}} \text{ and } \dot{\boldsymbol{z}}_{\boldsymbol{k+1}} = \boldsymbol{A}\,\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E}\,\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathrm{k+1}}}$$

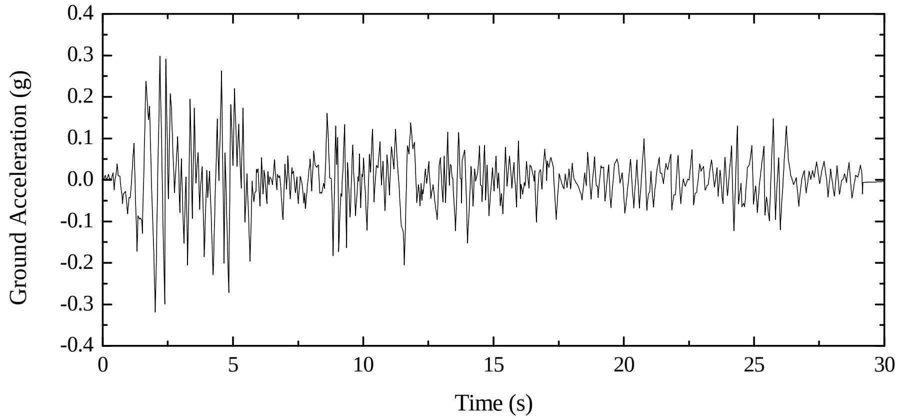
By considering the above calculated matrices and equations and assuming the initial displacements and velocities as zero, the response quantities for next time step are to be calculated. Table 5.1 gives the response for first 10 time steps for relative displacements and relative as well as absolute accelerations at super structure degrees of freedom. Note that $\ddot{x}^t$ in this table is the absolute (total) acceleration of the mass, which is equal to,

$$\ddot{\boldsymbol{x}}^{\boldsymbol{t}} = \ddot{\boldsymbol{x}} + \boldsymbol{1}\ddot{x}_g$$

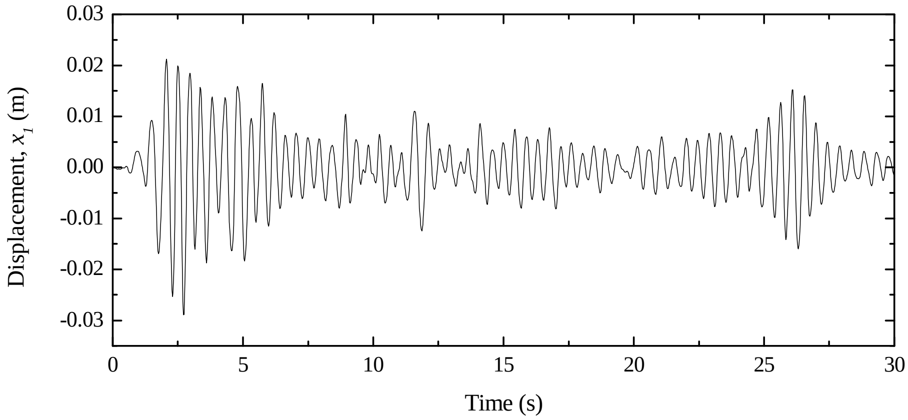
Table 5.1 Response of the system subjected to El-Centro earthquake

Time (s)	$\ddot{x}_g$ (m/s^2)	x_1 (m)	x_2 (m)	$\ddot{x}_1$ (m/s^2)	$\ddot{x}_2$ (m/s^2)	$\ddot{x}_1{}^t$ (m/s^2)	$\ddot{x}_2{}^t$ (m/s^2)
0.00	0.06180	0.00000	0.00000	-0.06180	-0.06180	0.00000	0.00000
0.02	0.03571	-0.00001	-0.00001	-0.02873	-0.03442	0.00698	0.00129
0.04	0.00971	-0.00004	-0.00004	0.00896	-0.00654	0.01868	0.00317
0.06	0.04199	-0.00007	-0.00008	-0.01362	-0.03489	0.02837	0.00710
0.08	0.07436	-0.00010	-0.00013	-0.03752	-0.05945	0.03684	0.01491
0.10	0.10663	-0.00014	-0.00019	-0.05884	-0.07999	0.04780	0.02665
0.12	0.06690	-0.00019	-0.00028	-0.00241	-0.02475	0.06449	0.04215
0.14	0.02717	-0.00026	-0.00039	0.05368	0.03295	0.08085	0.06013
0.16	-0.01256	-0.00031	-0.00049	0.10101	0.09259	0.08845	0.08003
0.18	0.03610	-0.00033	-0.00056	0.04527	0.06349	0.08138	0.09959
0.20	0.08476	-0.00032	-0.00060	-0.01749	0.03143	0.06727	0.11619

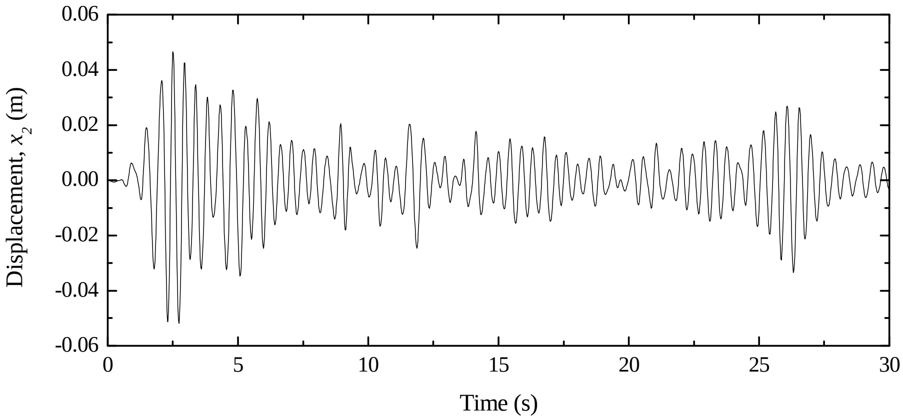
Extending the above sample calculations up to 30 sec gives the entire time histories for the responses. Figures 5.9 (a) and (b) show the time histories of relative displacements corresponding to the super structure degrees of freedom 1 and 2 and similarly, Figures 5.10 (a) and (b) show the time histories of absolute accelerations.



Figures 5.8 Acceleration time history of El-Centro, 1940 (N-S) earthquake (http://www.vibrationdata.com/elcentro.dat)

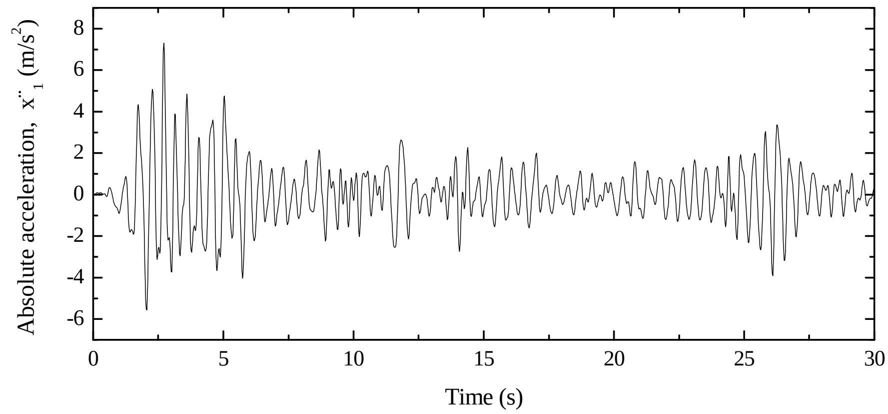


(a)

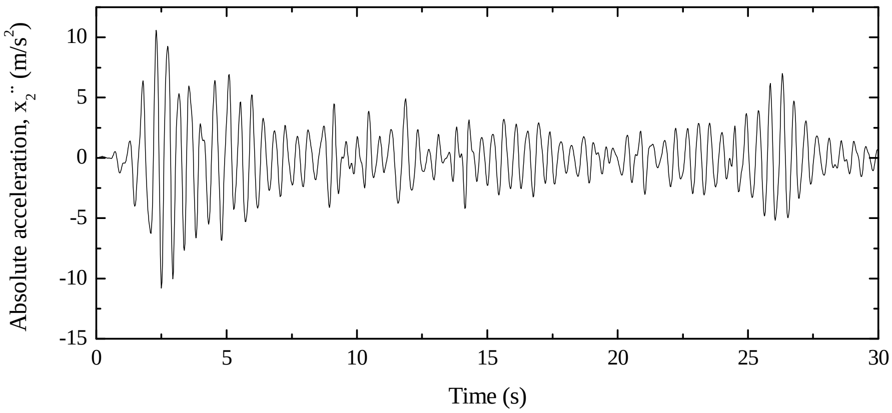


(b)

Figures 5.9 Time histories of relative displacements (a) displacement, x_1 ; and (b) displacement, x_2



(a)



(b)

Figures 5.10 Time histories of absolute accelerations (a) acceleration, $\ddot{x}_1{}^t$; and (b) acceleration, $\ddot{x}_2{}^t$

Case (ii) Considering the multi support excitations

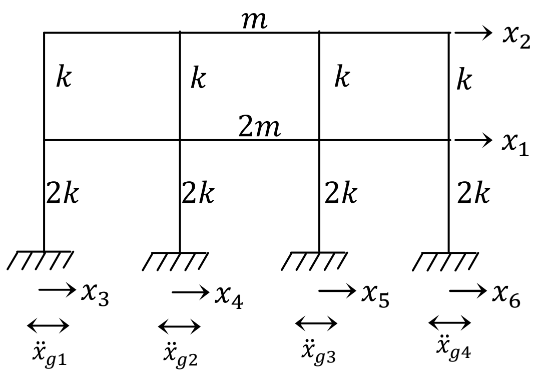


Figure 5.11 A Frame subjected to multi support excitation

For the system as shown in figure 5.11, the equations of motion can be written as follows,

$$\boldsymbol{M}_{ss}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}_{ss}\,\dot{\boldsymbol{x}} + \boldsymbol{K}_{ss}\,\boldsymbol{x} = -\boldsymbol{M}_{ss}\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

As derived earlier, the influence coefficient matrix, $\boldsymbol{\Gamma}$ can be calculated as follows,

$$\boldsymbol{\Gamma} = -\boldsymbol{K}_{ss}{}^{-1}\boldsymbol{K}_{sg} = -\begin{bmatrix} 12k & -4k \\ -4k & 4k \end{bmatrix}^{-1}\begin{bmatrix} -2k & -2k & -2k & -2k \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

For the given system, the four supports are subjected to earthquake excitation with time delay. Hence, there will be four component of acceleration in the vector of earthquake ground motion as follows.

$$\ddot{\boldsymbol{x}}_{\boldsymbol{g}} = \begin{Bmatrix} \ddot{x}_{g1} \\ \ddot{x}_{g2} \\ \ddot{x}_{g3} \\ \ddot{x}_{g4} \end{Bmatrix}$$

Hence, the equations of motion for this system with multi support excitation can be written as follows,

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + [a_o\,\boldsymbol{M} + \, a_1\,\boldsymbol{K}] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 12k & -4k \\ -4k & 4k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= - \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{g1} \\ \ddot{x}_{g2} \\ \ddot{x}_{g3} \\ \ddot{x}_{g4} \end{Bmatrix}$$

<u>Considering the Time Delay Effect in Earthquake Ground Motion:</u>

The given system is subjected to El-Centro earthquake ground motion of total duration of 30 s. The time delay between two supports is given as 5 s, the total duration of earthquake records for $\ddot{x}_{g1}$, $\ddot{x}_{g2}$, $\ddot{x}_{g3}$ and $\ddot{x}_{g4}$ is to be considered as 45 s with details for individuals as follows.

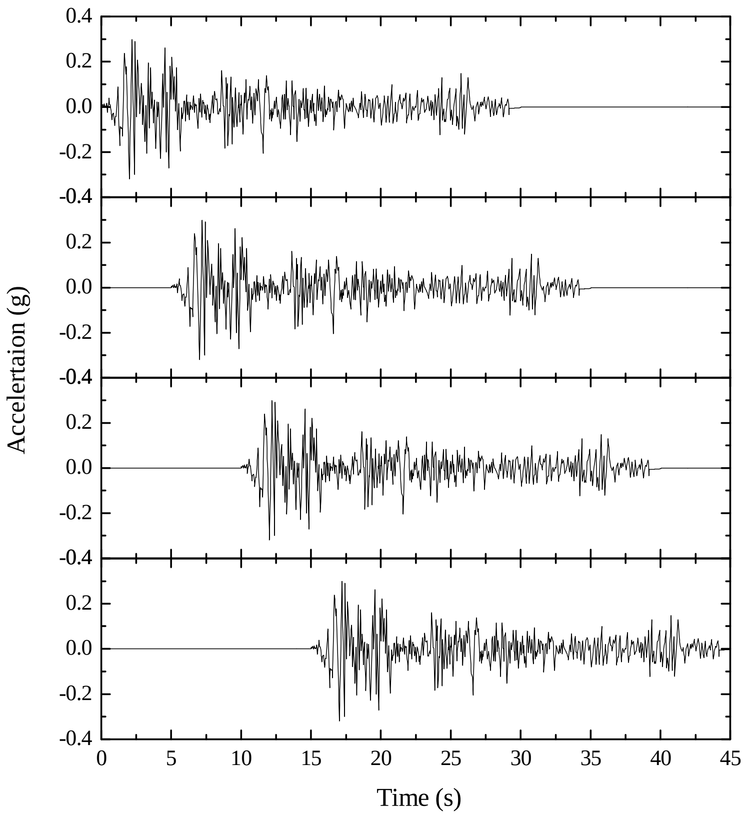
for $\ddot{x}_{g1}$:
The record of $\ddot{x}_{g1}$ will have, the first 30 s as the actual El Centro record and the last 15 s of the record will consists zeros.

for $\ddot{x}_{g2}$:
The record of $\ddot{x}_{g2}$ will have, the first 5 s record values as zeros followed by 30 s of the actual El Centro record and the last 10 s of the record will consists again the zeros.

for $\ddot{x}_{g3}$:
The record of $\ddot{x}_{g3}$ will have, the first 10 s record values as zeros followed by 30 s of the actual El Centro record and the last 5 s of the record will consists again the zeros.

for $\ddot{x}_{g4}$:
The record of $\ddot{x}_{g4}$ will have, the first 15 s record values as zeros followed by 30 s of the actual El Centro record as last values.

Now, the system is subjected to multi support El-Centro earthquake ground motion as shown in Figure 5.12, hence as discussed earlier, $\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$ will be having size of '4x1'. The time step, Δt is considered as 0.02 s. Assume k = 1000 *N/m* and m = 10 *kg*. Using the state space method with constant forcing function method, the solution of equations of motion can be obtained as follows.



Figures 5.12 Acceleration time history of El-Centro, 1940 earthquake for four supports with time delay of 5 s between supports

The matrices $\boldsymbol{A}$ and $\boldsymbol{A_d}$ will be exactly same as those obtained for the case (i) (i.e. for single support excitation).

$$\boldsymbol{E_d} = \boldsymbol{A}^{-1}(\boldsymbol{A_d} - \boldsymbol{I})\boldsymbol{E} \text{ , where } \boldsymbol{E} = \begin{Bmatrix} \mathbf{0_{nx1}} \\ -\boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}_{\mathbf{nx1}}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.25\,(\ddot{x}_{g1} + \ddot{x}_{g2} + \ddot{x}_{g3} + \ddot{x}_{g4} \\ -0.25\,(\ddot{x}_{g1} + \ddot{x}_{g2} + \ddot{x}_{g3} + \ddot{x}_{g4} \end{Bmatrix}$$

$$\boldsymbol{z}_{k+1} = \boldsymbol{A_d}\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E_d} \text{ and } \dot{\boldsymbol{z}}_{\boldsymbol{k+1}} = \boldsymbol{A}\,\boldsymbol{z}_{\mathrm{k}} + \boldsymbol{E}$$

By considering the above equations and assuming the initial displacements and velocities as zero, the response quantities for next time step are to be calculated. Table 5.2 gives the response for first 10 time steps for relative displacements and relative as well as absolute accelerations at super structure degrees of freedom. Note that $\ddot{x}^t$ in this table is the absolute (total) acceleration of the mass, which is equal to,

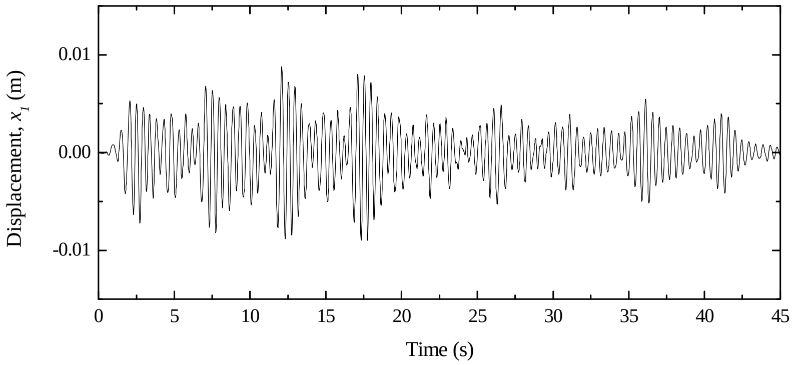
$$\ddot{\boldsymbol{x}}^{\boldsymbol{t}} = \ddot{\boldsymbol{x}} + \boldsymbol{\Gamma}\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

$$= \ddot{\boldsymbol{x}} + 0.25(\ddot{x}_{g1} + \ddot{x}_{g2} + \ddot{x}_{g3} + \ddot{x}_{g4})$$

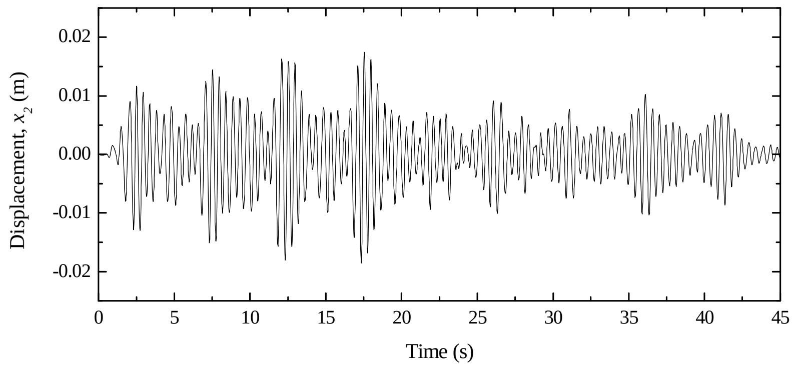
Table 5.2 Response of the system subjected to El-Centro earthquake

Time (s)	$\ddot{x}_{g1}$ (m/s^2)	$\ddot{x}_{g2} = \ddot{x}_{g3} = \ddot{x}_{g4}$ (m/s^2)	x_1 (m)	x_2 (m)	$\ddot{x}_1$ (m/s^2)	$\ddot{x}_2$ (m/s^2)	$\ddot{x}_1{}^t$ (m/s^2)	$\ddot{x}_1{}^t$ (m/s^2)
0.00	0.06180	0.00000	0.00000000	0.00000000	-0.01545	-0.01545	0.00000	0.00000
0.02	0.03571	0.00000	-0.00000301	-0.00000307	-0.00718	-0.00861	0.00174	0.00032
0.04	0.00971	0.00000	-0.00001017	-0.00001089	0.00224	-0.00163	0.00467	0.00079
0.06	0.04199	0.00000	-0.00001776	-0.00002064	-0.00341	-0.00872	0.00709	0.00177
0.08	0.07436	0.00000	-0.00002509	-0.00003224	-0.00938	-0.01486	0.00921	0.00373
0.10	0.10663	0.00000	-0.00003453	-0.00004812	-0.01471	-0.02000	0.01195	0.00666
0.12	0.06690	0.00000	-0.00004818	-0.00007035	-0.00060	-0.00619	0.01612	0.01054
0.14	0.02717	0.00000	-0.00006407	-0.00009703	0.01342	0.00824	0.02021	0.01503
0.16	-0.01256	0.00000	-0.00007667	-0.00012239	0.02525	0.02315	0.02211	0.02001
0.18	0.03610	0.00000	-0.00008129	-0.00014048	0.01132	0.01587	0.02034	0.02490
0.20	0.08476	0.00000	-0.00007899	-0.00014981	-0.00437	0.00786	0.01682	0.02905

Further, extending the above sample calculations up to 45 sec gives the entire time histories for the responses. Figures 5.13 (a) and (b) show the time histories of relative displacements corresponding to the super structure degrees of freedom 1 and 2 and similarly, Figures 5.14 (a) and (b) show the time histories of absolute accelerations.

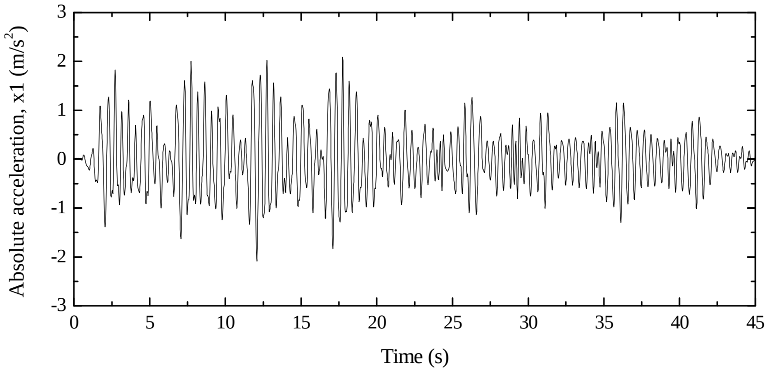


(a)

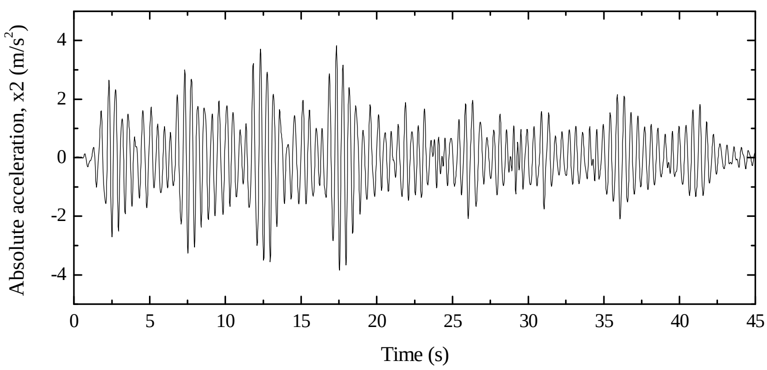


(b)

Figures 5.13 Time histories of relative displacements (a) displacement, x_1 ; and (b) displacement, x_2



(a)



(b)

Figures 5.14 Time histories of absolute accelerations (a) acceleration, $\ddot{x}_1{}^t$; and (b) acceleration, $\ddot{x}_2{}^t$

Exercise Problem

Example 1: For the portal frame as shown in Figure 5.15, calculate the peak and RMS values of relative displacements and absolute accelerations corresponding to the super structure degrees of freedom. (i.e. x_1 and x_2) when subjected to El-Centro earthquake ground motion for the following cases.
Case (i) Considering the same excitation at all supports (uniform excitation)
Case (ii) Considering multi-support excitations with a time delay of 5 s between supports
Assume percentage of critical damping as 5 %, k = 2000 N/m and m = 50.

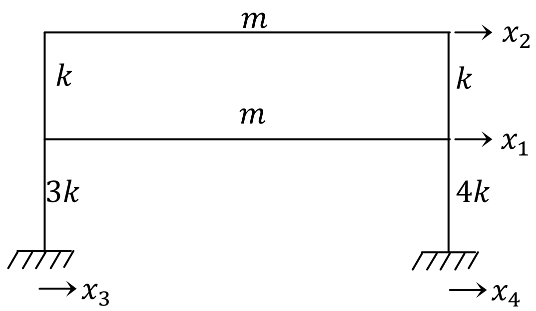


Figure 5.15

Answer 1:

Case (i)

Response quantities	Peak values	RMS values
Relative displacement at level 1, x_1 (m)	0.0291	0.0077
Relative displacement at level 2, x_2 (m)	0.1073	0.0139
Absolute acceleration at level 1, $\ddot{x}_1$ (m/s^2)	6.0421	1.1364
Absolute acceleration at level 2, $\ddot{x}_2$ (m/s^2)	6.4195	1.6229

Case (ii)

Response quantities	Peak values	RMS values
Relative displacement at level 1, x_1 (m)	0.0191	0.0051
Relative displacement at level 2, x_2 (m)	0.0622	0.0094
Absolute acceleration at level 1, $\ddot{x}_1$ (m/s^2)	3.3174	0.6371
Absolute acceleration at level 2, $\ddot{x}_2$ (m/s^2)	3.7429	1.0894