Chapter 3

Dynamics of Earthquake Analysis

3.1 Introduction

Earthquake or seismic analysis is a subset of structural analysis which involves the calculation of the response of a structure subjected to earthquake excitation. This is required for carrying out the structural design, structural assessment and retrofitting of the structures in the regions where earthquakes are prevalent. Various seismic data are necessary to carry out the seismic analysis of the structures. These data are accessible into two ways viz. in deterministic form or in probabilistic form. Data in deterministic form are used for design of structures etc whereas data in probabilistic form are used for seismic risk analysis, study of structure subjected to random vibration and damage assessment of structures under particular earthquake ground motion. Major seismic input includes ground acceleration/velocity/displacement data, magnitude of earthquake, peak ground parameters, duration etc.

In this chapter, the seismic response of the structures is investigated under earthquake excitation expressed in the form of time history of acceleration. The response is investigated for the structures modeled as Single Degree of Freedom (SDOF) and discrete Multi Degree of Freedom (MDOF) System.

3.2 Equation of Motion for SDOF System

Consider a SDOF system (shown in Figure 3.1), subjected to an earthquake acceleration, $\ddot{x}_g(t)$. Let *m*, *k* and *c* represent the mass, stiffness and damping, respectively of the SDOF system undergoing relative displacement, velocity and acceleration of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$, respectively. The various forces acting on the system will be inertial force, stiffness force and damping force.

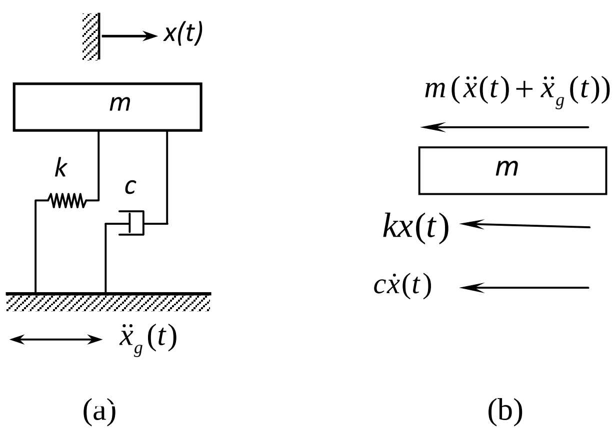

Figure 3.1 (a) SDOF system (b) Free body diagram.

Consider the equilibrium of the various forces acting on the mass, as shown in Figure 3.1(b), we get,

$$m(\ddot{x}(t) + \ddot{x}_g(t)) + c\dot{x}(t) + kx(t) \;=\; 0 \tag{3.1}$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) \;=\; -m\ddot{x}_g(t) \tag{3.2}$$

where,

$x(t)$ = relative displacement of mass with respect to ground

$\dot{x}(t)$ = relative velocity of mass with respect to ground

$\ddot{x}(t)$ = relative acceleration of mass with respect to ground

$\ddot{x}_g(t)$ = earthquake ground acceleration

The equation of motion is expressed in the normalized form as

$$\ddot{x}(t) \;+\; 2\xi\omega_0\dot{x}(t) \;+\; \omega_0^2\, x(t) \;=\; -\ddot{x}_g(t) \tag{3.3}$$

where, ξ and ω_0 denotes the damping ratio and natural frequency of SDOF system, respectively expressed as

$$\omega_0 \;= \sqrt{\frac{k}{m}} \tag{3.4}$$

$$\xi \;=\; \frac{c}{2m\omega_0} \tag{3.5}$$

The damped natural frequency of SDOF system is given by

$$\omega_d \;=\; \omega_0\sqrt{1-\xi^2} \tag{3.6}$$

The equation of motion for a linear, viscously damped SDOF system is second order differential equation with constant coefficients. The solution of this equation for the specified earthquake acceleration, $\ddot{x}_g(t)$ will provide the response of the SDOF system.

3.3 Response Analysis of SDOF System

For a given time history (acceleration versus time data) of earthquake ground motion, the response of viscously damped SDOF system can be obtained either by Time Domain Analysis or Frequency Domain Analysis.

3.3.1 Time Domain Analysis

This method helps in obtaining response of SDOF system in both linear and non linear range. Duhamel integration and Numerical schemes such as Newmark integration, Runge-Kutta methods are invariably accompanied for obtaining numerical solution of differential equation.

Duhamel Integral is used to obtain the response of SDOF system subjected to earthquake ground motion. Equation of motion for a SDOF system subjected to ground motion acceleration is given by equation (3.2). The solution of which can be split into homogeneous and particular part as

$$x(t) = x_h(t) + x_p(t) \tag{3.7}$$

where,

$x_h(t)$ = homogeneous solution, and $x_p(t)$ = particular solution.

Homogeneous or complimentary solution (as depicted from Figure 3.2) is the damped free-vibration response given by equation (3.8)

$$x_h(t) = g(t)\,x_0 + h(t)\,\dot{x}_0 \tag{3.8}$$

where, x_0 and $\dot{x}_0$ are initial displacement and velocity of the SDOF system, respectively.

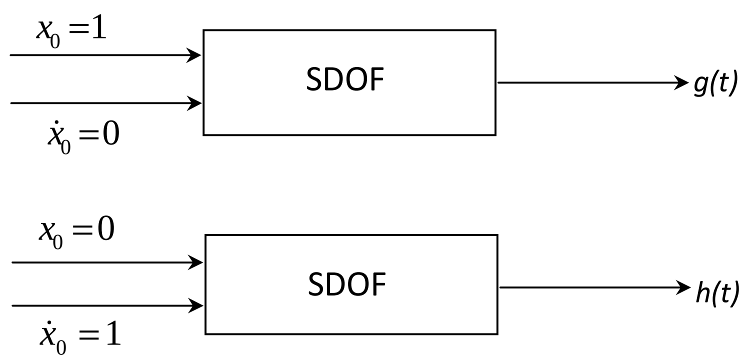

Figure 3.2 Response of SDOF system to initial unit displacement and velocity.

Putting the boundary conditions (as shown in Figure 3.2) in the solution of the homogeneous part, g(t) and $h($t) can be obtained as

$$g(t) = e^{-\xi\omega_0 t}\left[\cos\ \omega_d t + \frac{\xi\omega_0}{\omega_d}\sin\ \omega_d t\right] \tag{3.9}$$

$$h(t) = \frac{e^{-\xi\omega_0 t}}{\omega_d}\sin\ \omega_d t \tag{3.10}$$

For obtaining particular solution part of equation (3.7), it is assumed that the irregular ground acceleration is made up of very brief impulses as shown in Figure 3.3. The vibration caused by all the impulse are added together to obtain the total response.

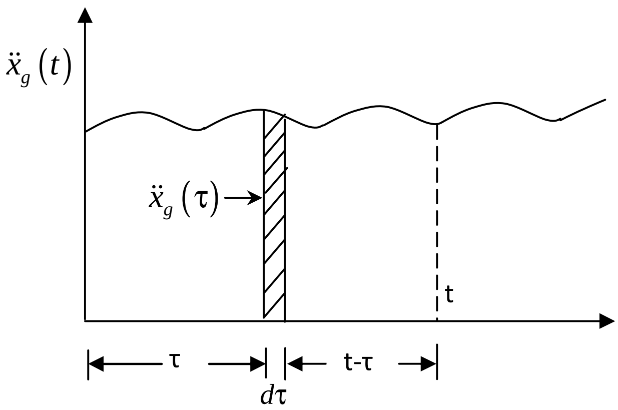

Figure 3.3 Impulse from earthquake acceleration.

Consider the vibration caused by a single impulse. Newton’s second law states that the rate of change of moment of a mass is equal to the applied force i.e.

$$\frac{d}{dt}(m\dot{x}(t)) = -m\,\ddot{x}_g(t) \tag{3.11}$$

Thus, the change in momentum over a brief interval, $d\tau$ brought by the instantaneous force $-m\ \ddot{x}_g(\tau)$ is given by

$$d(m\dot{x}(t)) = -m\ \ddot{x}_g(\tau)\ d\tau \tag{3.12}$$

It should be noted that the small changes in velocity and displacement occurring during the time interval dτ will make a negligible contribution to the change in momentum. The change in velocity during the interval is

$$d\dot{x}(t) \;=\; -\;\ddot{x}_g(\tau)\;d\tau \tag{3.13}$$

Thus, the change in displacement at time, t caused by the impulse at τ is given by

$$dx_p(t) \;=\; -\;\ddot{x}_g(\tau)\;d\tau.\;h(t-\tau) \tag{3.14}$$

Each impulse in Figure (3.3) will produce a vibration of this form. Because the system is linear, the effect of each impulse is independent of every other impulse and the total resulting motion can be obtained by the principle of super position.

$$x_p(t) \;=\; -\int_0^t \ddot{x}_g(\tau)\;\; h(t-\tau)\;d\tau \tag{3.15}$$

This integral is known as **convolution or Duhamel integral**. Explicit solution may be obtained for simple forms of forcing function such as rectangular and triangular.

From equations (3.8), (3.9), (3.10) and (3.15), the total response (given in equation (3.7)) of system can be given by

$$x(t) = x_0 g(t) + \dot{x}_0 h(t) - \int_0^t [\ddot{x}_g(\tau)] h(t-\tau) d\tau \tag{3.16}$$

For the system with at rest condition (i.e. x_0 =0 and $\dot{x}_0$ =0) the response is given by

$$x(t) \;=\; -\int_0^t \ddot{x}_g(\tau)\;\; h(t-\tau)\;d\tau \tag{3.17}$$

This is known as time domain solution because the response is calculated using time as a variable.

In order to obtain recurrence formulas for time domain analysis, consider a SDOF system with displacement and velocity defined at initial time, t_i and the response is required at, t_{i+1} (refer Figure 3.4). Suppose x_i and $\dot{x}_i$ are the initial displacement and velocity of the system, respectively,

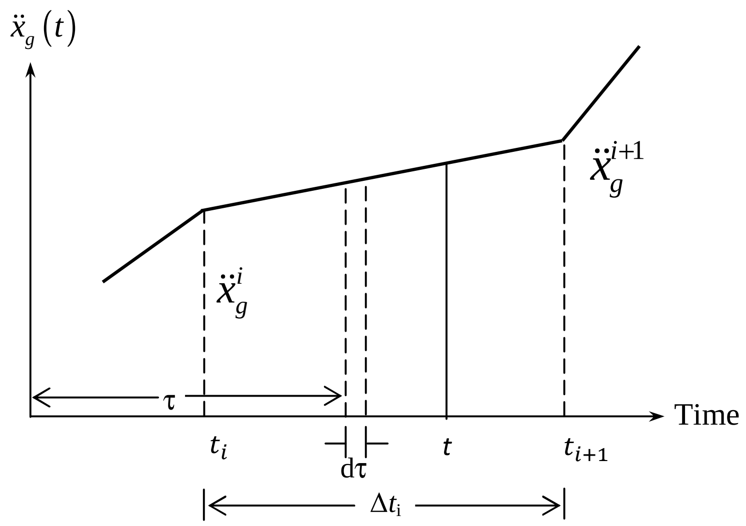

Figure 3.4 Linear variation of ground acceleration across chosen time step Δt_i .

The response of the system (reproducing equation (3.16)) is expressed as

$$x(t) = g\,(t-t_i)\,x_i + h\,(t-t_i)\,\dot{x}_i - \int_{t_i}^{t} \ddot{x}_g(\tau)\; h\,(t-\tau)\, d\tau \tag{3.18}$$

$$\left.\begin{aligned}
g(t-t_i) &= e^{-\xi\omega_0(t-t_i)}\left\{\frac{\xi\omega_0}{\omega_d}\ \sin\ \omega_d(t-t_i) + \cos\ \omega_d(t-t_i)\right\} \\
h(t-t_i) &= \frac{e^{-\xi\omega_0(t-t_i)}}{\omega_d}\sin\ \omega_d(t-t_i) \\
\ddot{x}_g(\tau) &= \ddot{x}_g^{\,i} + \frac{\ddot{x}_g^{\,i+1}-\ddot{x}_g^{\,i}}{\Delta t_i}\ (\tau - t_i) \\
h(t-\tau) &= \frac{e^{-\xi\omega_0(t-\tau)}}{\omega_d}\ \sin\ \omega_d(t-\tau)
\end{aligned}\right\} \tag{3.19}$$

Back substituting in equation (3.18),

$$x(t) = g(t-t_i)x_i + h(t-t_i)\dot{x}_i - \int_{t_i}^{t}\left[\ddot{x}_g^i + \frac{\ddot{x}_g^{i+1}-\ddot{x}_g^{\,i}}{\Delta t_i}(\tau-t_i)\right]\left\{\frac{e^{-\xi\omega_0(t-\tau)}}{\omega_d}\ \sin\ \omega_d(t-\tau)\right\}d\tau \tag{3.20}$$

$$x(t) = g(t-t_i)x_i + h(t-t_i)\dot{x}_i + f_1(t-t_i)\,\ddot{x}_g^i + f_2(t-t_i)\,\ddot{x}_g^{i+1} \tag{3.21}$$

Similarly, the velocity of the system at time, t is given by

$$\dot{x}(t) = \dot{g}(t-t_i)x_i + \dot{h}(t-t_i)\dot{x}_i + \dot{f}_1(t-t_i)\ddot{x}_g^i + \dot{f}_2(t-t_i)\ddot{x}_g^{i+1} \tag{3.22}$$

At $t = t_{i+1}$, $x(t) = x_{i+1}$ and $\dot{x}(t) = \dot{x}_{i+1}$, the displacement and velocity of the system are expressed as

$$x_{i+1} = g(\Delta t_i)x_i + h(\Delta t_i)\dot{x}_i + f_1(\Delta t_i)\ddot{x}_g^i + f_2(\Delta t_i)\ddot{x}_g^{i+1} \tag{3.23}$$

$$\dot{x}_{i+1} = \dot{g}(\Delta t_i)x_i + \dot{h}(\Delta t_i)\dot{x}_i + \dot{f}_1(\Delta t_i)\ddot{x}_g^i + \dot{f}_2(\Delta t_i)\ddot{x}_g^{i+1} \tag{3.24}$$

In the matrix form, the above equations can be re-written

$$\begin{Bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{Bmatrix} = \begin{bmatrix} g(\Delta t_i) & h(\Delta t_i) \\ \dot{g}(\Delta t_i) & \dot{h}(\Delta t_i) \end{bmatrix} \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix} + \begin{bmatrix} f_1(\Delta t_i) & f_2(\Delta t_i) \\ \dot{f}_1(\Delta t_i) & \dot{f}_2(\Delta t_i) \end{bmatrix} \begin{Bmatrix} \ddot{x}_g^i \\ \ddot{x}_g^{i+1} \end{Bmatrix} \tag{3.25}$$

or

$$\begin{Bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_g^i \\ \ddot{x}_g^{i+1} \end{Bmatrix} \tag{3.26}$$

or

$$\{\bar{x}_{i+1}\} = [A]\{\bar{x}_i\} + [B]\{\ddot{\bar{x}}_g^{\,i}\} \tag{3.27}$$

where,

$$\{\bar{x}_i\} = \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix}, \quad \{\ddot{\bar{x}}_g^i\} = \begin{Bmatrix} \ddot{x}_g^i \\ \ddot{x}_g^{i+1} \end{Bmatrix}, \quad [A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{and} \quad [B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The elements of matrices [A] and [B] from Nigam and Jennings (1969) are given by equations (3.28) and (3.29)

$$
\left.\begin{aligned}
a_{11} &= e^{-\xi\omega_0\Delta t_i}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\ \omega_d\Delta t_i\ +\ \cos\ \omega_d\Delta t_i\right)\\
a_{12} &= \frac{e^{-\xi\omega_0\Delta t_i}}{\omega_d}\ \sin\ \omega_d\Delta t_i\\
a_{21} &= -\ \frac{\omega_0}{\sqrt{1-\xi^2}}\ e^{-\xi\omega_0\Delta t_i}\ \ \sin\ \omega_d\Delta t_i\\
a_{22} &= e^{-\xi\omega_0\Delta t_i}\ \left(\cos\ \omega_d\Delta t_i\ -\ \frac{\xi}{\sqrt{1-\xi^2}}\ \sin\ \omega_d\Delta t_i\right)
\end{aligned}\right\} \tag{3.28}
$$

$$
\left.\begin{aligned}
b_{11} &= e^{-\xi\omega_0\Delta t_i}\left[\left(\frac{2\xi^2-1}{\omega_0^2\,\Delta t_i}+\frac{\xi}{\omega_0}\right)\frac{\sin\ \omega_d\Delta t_i}{\omega_d}+\left(\frac{2\xi}{\omega_0^3\,\Delta t_i}+\frac{1}{\omega_0^2}\right)\cos\ \omega_d\Delta t_i\right]\\
b_{12} &= e^{-\xi\omega_0\Delta t_i}\left[\left(\frac{2\xi^2-1}{\omega_0^2\,\Delta t_i}\right)\frac{\sin\ \omega_d\Delta t_i}{\omega_d}\ +\ \frac{2\xi}{\omega_0^3\,\Delta t_i}\cos\ \omega_d\Delta t_i\right]-\frac{1}{\omega_0^2}+\frac{2\xi}{\omega_0^3\,\Delta t_i}\\
b_{21} &= e^{-\xi\omega_0\Delta t_i}\left[\left(\frac{2\xi^2-1}{\omega_0^2\,\Delta t_i}+\frac{\xi}{\omega_0}\right)\left(\cos\ \omega_d\Delta t_i-\frac{\xi}{\sqrt{1-\xi^2}}\sin\ \omega_d\Delta t_i\right)\right.\\
&\qquad\left.-\left(\frac{2\xi}{\omega_0^3\,\Delta t_i}+\frac{1}{\omega_0^2}\right)\left(\omega_d\sin\ \omega_d\Delta t_i+\xi\omega_0\cos\ \omega_d\Delta t_i\right)\ \right]+\ \frac{1}{\omega_0^2\,\Delta t_i}\\
b_{22} &= e^{-\xi\omega_0\Delta t_i}\left[\frac{2\xi^2-1}{\omega_0^2\,\Delta t_i}\left(\cos\ \omega_d\Delta t_i-\frac{\xi}{\sqrt{1-\xi^2}}\sin\ \omega_d\Delta t_i\right)\right.\\
&\qquad\left.-\frac{2\xi}{\omega_0^3\,\Delta t_i}\left(\omega_d\sin\ \omega_d\Delta t_i+\xi\omega_0\cos\ \omega_d\Delta t_i\right)\right]-\frac{1}{\omega_0^2\,\Delta t_i}
\end{aligned}\right\} \tag{3.29}
$$

Finally, the acceleration response of the SDOF system can be obtained by reproducing the equation of motion as

$$\ddot{x}_{i+1} = \frac{(-c\dot{x}_{i+1} - kx_{i+1} - m\ddot{x}_g^{i+1})}{m} \tag{3.30}$$

Hence, if the displacement and velocity of the system are known at some time t_i, the state of the system at all subsequent times, t_{i+1} can be computed exactly by a step-by-step application of equation (3.27) to (3.30). The computational advantage of this approach lies in the fact that the elements of [*A*] and [*B*] matrix depend only on ξ, ω_0 and Δt_i. The value ξ and ω_0 are constant and if Δt_i is also constant, x_i , $\dot{x}_i$ *and* $\ddot{x}_i$ can be evaluated by the execution of multiplication and summation operations for each step of integration. The matrices [*A*] and [*B*], defined by rather complicated expressions, equations (3.28) and (3.29) need to be evaluated only at the beginning of each response evaluation. If varying time intervals are used, it is necessary, in general, to compute [*A*] and [*B*] at each step of integration. However, by rounding the time coordinates of the record, the number of these matrices needed during the calculation can be reduced to only a few. These, too can be computed at the beginning of the calculation and called upon when needed, thereby saving the computational time.

3.3.1.1 Numerical Methods for Seismic Analysis of SDOF System

There are number of numerical methods available for solving initial boundary value problems. Most commonly used methods are Newmark's Beta method (Linear acceleration method) and Runge-Kutta method which are described here.

3.3.1.1.1 Newmark's Beta Method

In this method, acceleration, velocity and displacement at time, $t = t_{i+1}$ is obtained as a function of acceleration, velocity and displacement at $t = t_i$ (which is always known), assuming linear acceleration during small time step (Figure 3.5). Assume a SDOF system subjected to earthquake ground motion, the equation of motion is given by

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -m\ddot{x}_g(t) \tag{3.31}$$

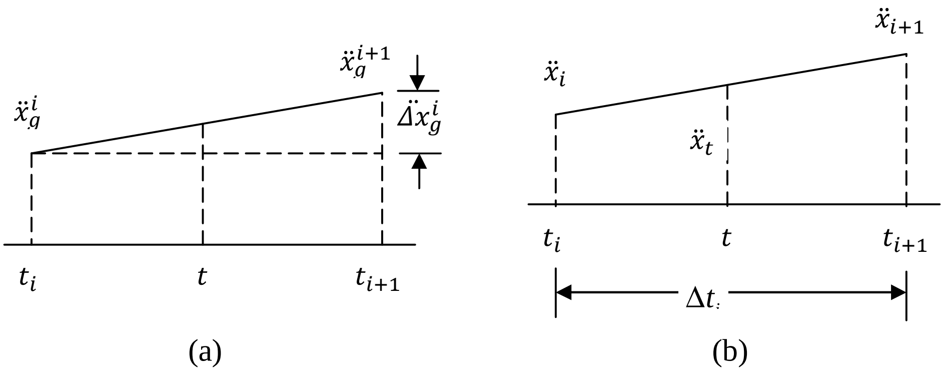

Figure 3.5 (a) Linear ground acceleration, (b) Linear relative acceleration of SDOF system over time step, Δt_i .

In the incremental form, equation (3.31) can be re-written as

$$m\Delta\ddot{x}_i + c\Delta\dot{x}_i + k\Delta x_i = -m\Delta\ddot{x}_g^i \tag{3.32}$$

where,

$$\left.\begin{aligned} \Delta\ddot{x}_i &= \ddot{x}_{i+1} - \ddot{x}_i \\ \Delta\dot{x}_i &= \dot{x}_{i+1} - \dot{x}_i \\ \Delta x_i &= x_{i+1} - x_i \\ \Delta t_i &= t_{i+1} - t_i \\ \Delta\ddot{x}_g^i &= \ddot{x}_g^{i+1} - \ddot{x}_g^i \end{aligned}\right\} \tag{3.33}$$

Assuming linear variation of acceleration (see Figure 3.5(b))

$$\ddot{x}(t) = \ddot{x}_i + \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t_i} \cdot t \tag{3.34}$$

On integrating,

$$\dot{x}(t) = \dot{x}_i + \ddot{x}_i\, t + \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t_i} \cdot \frac{t^2}{2} \tag{3.35}$$

$$x(t) = x_i + \dot{x}_i\, t + \ddot{x}_i \frac{t^2}{2} + \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t_i} \cdot \frac{t^3}{6} \tag{3.36}$$

In equation (3.36) put $t = \Delta t_i$ and express in terms of $\Delta\ddot{x}_i$ i.e.

$$\Delta\ddot{x}_i = \frac{6}{\Delta t_i^2}\Delta x_i - \frac{6}{\Delta t_i}\dot{x}_i - 3\ddot{x}_i \tag{3.37}$$

Similarly, from equation (3.35) put $t = \Delta t_i$ and solve for $\Delta\dot{x}_i$

$$\Delta\dot{x}_i = \frac{3}{\Delta t_i}\Delta x_i - 3\dot{x}_i - \frac{\Delta t_i}{2}\ddot{x}_i \tag{3.38}$$

Substituting $\Delta\ddot{x}_i$, $\Delta\dot{x}_i$ in equation (3.32) and solve for Δx_i i.e.

$$\Delta x_i = \frac{p_{eff}}{k_{eff}} \tag{3.39}$$

where,

$$p_{eff} = -m\Delta\ddot{x}_g^{\,i} + \left(\frac{6}{\Delta t_i}m + 3c\right)\dot{x}_i + \left(3m + \frac{\Delta t_i}{2}c\right)\ddot{x}_i \tag{3.40}$$

$$k_{eff} = \frac{6}{\Delta t_i^{\,2}}m + \frac{3}{\Delta t_i}c + k \tag{3.41}$$

Knowing the Δx_i, determine $\Delta\dot{x}_i$ from equation (3.38). At $t = t_{i+1}$, displacement and velocity can be determined as

$$\left.\begin{aligned} x_{i+1} &= x_i + \Delta x_i \\ \dot{x}_{i+1} &= \dot{x}_i + \Delta\dot{x}_i \end{aligned}\right\} \tag{3.42}$$

The acceleration at time t_{i+1} is calculated by considering equilibrium of equation (3.31) to avoid the accumulation of the unbalanced forces i.e.

$$\ddot{x}_{i+1} = \frac{1}{m}\left[-m\ddot{x}_g^{i+1} - c\dot{x}_{i+1} - kx_{i+1}\right] \tag{3.43}$$

In this way, using this step-by-step numerical integration scheme, the response of SDOF system can be obtained for given time history. Repeat the same steps to obtain response at $t = t_{i+2}$ and so on. The accuracy of output response depends upon the magnitude of time-step 'Δt_i' chosen. Optimum values of 'Δt_i' should be chosen to obtain fastest converging results with required precision. Time stepping methods has got limitation that error goes on accumulating with calculation proceeds. In order to keep Newmark's Beta method stable, the time step should be taken such that (Chopra, 2007)

$$\frac{\Delta t_i}{T_0} \leq 0.551 \tag{3.44}$$

where, $T_0 = 2\pi / \omega_0$,is the time period of the SDOF system.

This method is conditionally stable if above inequality satisfies, otherwise method will "blow-up" giving illogical results.

3.3.1.1.2 Runge-Kutta Method

Knowing the initial conditions, response of SDOF system with time can be determined using Runge-Kutta method. Let the equation of motion of the SDOF system be

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -m\ddot{x}_g(t) \tag{3.45}$$

Define a vector,

$$\overline{x} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{3.46a}$$

On differentiating the above equation

$$\dot{\bar{x}} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} \tag{3.46b}$$

Using Equation (3.45) and (3.46a), above equation can be reproduced as,

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\ddot{x}_g \end{Bmatrix}$$

or, $$\dot{\bar{x}} = \bar{E}\bar{x} + \bar{F} \tag{3.47}$$

where, $\bar{E} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix}$; $\bar{F} = \begin{Bmatrix} 0 \\ -\ddot{x}_g \end{Bmatrix}$

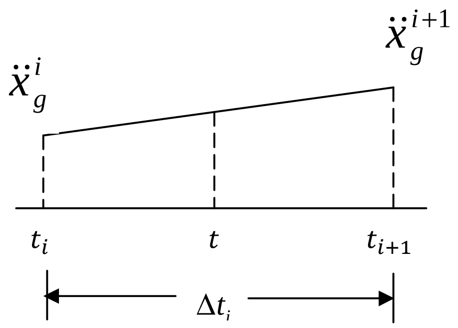

Figure 3.6: Time step for Runge-Kutta method

Determine the following constant vectors,

$$\bar{k}_1 = \Delta t_i \left[\bar{E}\bar{x}_i + \bar{F}_i \right] \tag{3.48}$$

$$\bar{k}_2 = \Delta t_i \left[\bar{E}\left(\bar{x}_i + \frac{\bar{k}_1}{3} \right) + \left(\bar{F}_i + \frac{\Delta \bar{F}_i}{3} \right) \right] \tag{3.49}$$

$$\bar{k}_3 = \Delta t_i \left[\bar{E}\left(\bar{x}_i + \frac{2\bar{k}_2}{3} \right) + \left(\bar{F}_i + \frac{2\Delta \bar{F}_i}{3} \right) \right] \tag{3.50}$$

(Note: Subscript 'i' refers to the value at time $t = t_i$, 'Δ' refers to difference in value at t_{i+1} and t_i, and '¯' indicates a matrix)

then,

$$\Delta\bar{x}_i = \begin{Bmatrix} \Delta x_i \\ \Delta\dot{x}_i \end{Bmatrix} = \frac{\bar{k}_1 + 2\bar{k}_2 + \bar{k}_3}{4} \tag{3.51}$$

$$\bar{x}_{i+1} = \begin{Bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{Bmatrix} = \bar{x}_i + \Delta\bar{x}_i \tag{3.52}$$

From equations (3.52) and (3.45)

$$\ddot{x}_{i+1} = \frac{(-c\dot{x}_{i+1} - kx_{i+1} - m\ddot{x}_g^{i+1})}{m} \tag{3.53}$$

Repeat same steps to obtain response at time $t = t_{i+2}$ and so on. A Runge-Kutta method is conditionally stable for linear second order differential equation like equation (3.45). But still the time step, Δt_i should be taken short enough to ensure required precision in results.

3.3.2 Frequency Domain Analysis

This method is used for obtaining response of linear systems subjected to irregular excitations such as earthquake forces and it requires knowledge of complex frequency response function for its proper application. If the stiffness '*k*' and damping '*c*' of the SDOF system are frequency dependent, then this approach is much superior to the time domain.

In frequency domain analysis, the response of a SDOF system is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ddot{X}_g(\omega) H(\omega) e^{i\omega t} d\omega \tag{3.54}$$

where, $\ddot{X}_g(\omega)$ is the Fourier transform of [- $\ddot{x}_g(t)$] and $H(\omega)$ is the complex frequency response function.

Consider a SDOF system (Figure 3.7), subjected to the forcing function of $e^{i\omega t}$, producing displacement response as

$$x(t) = H(\omega) e^{i\omega t} \tag{3.55}$$

Figure 3.7: Explanation of complex frequency response function.

Consider equation of motion of SDOF system

$$\ddot{x} + 2\xi\omega_o \dot{x} + \omega_o^2 x = e^{i\omega t} \tag{3.56}$$

Substituting equation (3.55) in equation (3.56)

$$\mathrm{H}(\omega)(i\omega)^2 e^{i\omega t} + 2\xi\omega_o H(\omega) i\omega e^{i\omega t} + \omega_o^2 \mathrm{H}(\omega) e^{i\omega t} = e^{i\omega t} \tag{3.57}$$

or, $H(\omega)[-\omega^2 + 2\xi\omega\omega_o i + \omega_o^2] = 1$ (3.58)

Thus, the complex frequency response function is expressed as

$$H(\omega) = \frac{1}{(\omega_o^2 - \omega^2) + i2\xi\omega\omega_o} \tag{3.59}$$

The Fourier transform of [- $\ddot{x}_g(t)$] is expressed as

$$\ddot{X}_g(\omega) = \int_{-\infty}^{\infty} [-\ddot{x}_g(t)] e^{-i\omega t} dt \tag{3.60}$$

The response of a SDOF system can be obtained by substituting equations (3.59) and (3.60) in equation (3.54) and solving the integral.

Note: Properties of Fourier transform are as follows,

Consider a function $f(t)$ which is periodic and it is absolutely integrable i.e.,

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty \tag{3.61}$$

Then the Fourier transform of $f(t)$ exists and given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{3.62}$$

From the inverse Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{3.63}$$

The $f(t)$ and $F(\omega)$ makes the Fourier Pair.

3.4 Numerical Examples on SDOF system

Example 3.1

A SDOF system is subjected to a harmonic ground motion of $\ddot{x}_g(t) = \ddot{x}_o \sin \bar{\omega} t$. Determine the steady state response using time and frequency domain method and considering that the system starts from rest. The natural frequency and fraction of critical damping of SDOF system are ω_o and ξ, respectively.

Solution: Equation of motion is given by

$$\ddot{x}(t) + 2\xi\omega_o \dot{x}(t) + \omega_o^2 x(t) = -\ddot{x}_o \sin \bar{\omega} t$$

A. Time Domain Analysis (Using Duhamel Integration)

Using equation (3.17) steady state response of system in time domain is given by

$$x(t) = -\int_0^t \ddot{x}_g(\tau) h(t-\tau) d\tau$$

where, $\ddot{x}_g(\tau) = \ddot{x}_o \sin \bar{\omega}\tau$ and $h(t-\tau) = \dfrac{e^{-\xi\omega_o(t-\tau)}}{\omega_d} \sin(\omega_d (t-\tau))$

Thus,

$$x(t) = -\int_0^t (\ddot{x}_o \sin \bar{\omega}\tau) \frac{e^{-\xi\omega_o(t-\tau)}}{\omega_d} \sin(\omega_d(t-\tau)) d\tau$$

$$= -\frac{\ddot{x}_o}{\omega_o^2} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \sin(\bar{\omega} t - \theta)$$

where, $\beta = \dfrac{\bar{\omega}}{\omega_o}$ and $\theta = \tan^{-1}\left(\dfrac{2\xi\beta}{1-\beta^2}\right)$

B. Frequency Domain Analysis

The steady state response is given by equation (3.54) i.e.,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{X}_g(\omega) H(\omega) e^{i\omega t} d\omega$$

where,

$$H(\omega) = \frac{1}{\omega_o^2 - \omega^2 + i2\xi\omega\omega_o}$$

and

$$\ddot{X}_g(\omega) = \mathrm{FT}\left[-\ddot{x}_o \sin(\overline{\omega}t)\right] = \int_{-\infty}^{\infty} \left[-\ddot{x}_o \sin(\overline{\omega}t)\right] e^{-i\omega t} dt$$

Evaluating the integral,

$$\ddot{X}_g(\omega) = \mathrm{FT}\left[-x_o \sin(\overline{\omega}t)\right] = -\int_{-\infty}^{\infty} \ddot{x}_o \sin(\overline{\omega}t) e^{-i\omega t} dt$$

$$= -\frac{\ddot{x}_o}{2i} \int_{-\infty}^{\infty} \left(e^{i\overline{\omega}t} - e^{-i\overline{\omega}t}\right) e^{-i\omega t} dt \qquad \left(Note: \sin\overline{\omega}t = \frac{e^{i\overline{\omega}t} - e^{-i\overline{\omega}t}}{2i}\right)$$

$$= -\frac{\ddot{x}_o}{2i} \left[\int_{-\infty}^{\infty} e^{i(\overline{\omega}-\omega)t} dt - \int_{-\infty}^{\infty} e^{i(-\omega-\overline{\omega})t} dt\right]$$

$$= -\frac{\ddot{x}_o}{2i} \left[2\pi\delta(\overline{\omega}-\omega) - 2\pi\delta(-\omega-\overline{\omega})\right]$$

$$= i\pi\ddot{x}_o \left[\delta(\omega-\overline{\omega}) - \delta(\omega+\overline{\omega})\right]$$

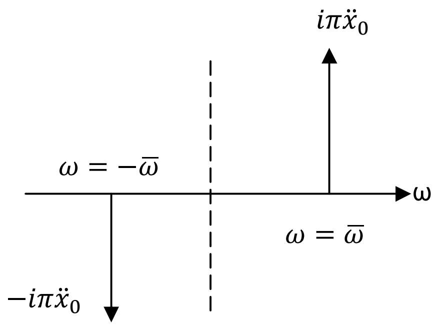

Figure 3.8 Graphical Representation of $\mathrm{FT}\,[-\ddot{x}_0 \sin\overline{\omega}t]$.

Recalling the properties of Fourier Transform,

$$1 = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \quad \text{and} \quad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1.e^{i\omega t} d\omega$$

Implying that,

$$\int_{-\infty}^{\infty} e^{i\omega t} d\omega = 2\pi\delta(t)$$

Recalling the properties of Dirac Delta ‘δ(t)’ function

$$\delta(\mathrm{kt}) = \frac{1}{|\mathrm{k}|}\delta(\mathrm{t}) \quad \text{and} \quad \delta(t - t_o) = 0 \ \text{ if } \ t \neq t_o$$

$$\int_{-\infty}^{\infty} \delta(t - t_o) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_o)\psi(t) dt = \psi(t_o)$$

$$\delta(t - t_o) = \delta(t_o - t)$$

Using the above properties, the response of the SDOF system is expressed as

$$x(t) = \frac{1}{2\pi}\int e^{i\omega t} H(\omega)\ddot{X}_g(\omega) d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left(\frac{1}{\omega_o^2 - \omega^2 + i2\xi\omega\omega_o}\right)\left(i\pi\ddot{x}_o\left[\delta(\omega - \overline{\omega}) - \delta(\omega + \overline{\omega})\right]\right) d\omega$$

$$= \frac{i\ddot{x}_o}{2}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_o^2 - \omega^2 + i2\xi\omega\omega_o}\delta(\omega - \overline{\omega}) d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_o^2 - \omega^2 + i2\xi\omega\omega_o}\delta(\omega + \overline{\omega}) d\omega\right]$$

$$= \frac{i\ddot{x}_o}{2}\left[\frac{e^{i\overline{\omega}t}}{\omega_o^2 - \overline{\omega}^2 + i2\xi\overline{\omega}\omega_o} - \frac{e^{-i\overline{\omega}t}}{\omega_o^2 - \overline{\omega}^2 - i2\xi\overline{\omega}\omega_o}\right]$$

Taking $\beta = \dfrac{\overline{\omega}}{\omega_o}$ and $\theta = \tan^{-1}\left(\dfrac{2\xi\beta}{1 - \beta^2}\right)$,

$$= \frac{i\ddot{x}_o}{2}\left[\frac{e^{i\overline{\omega}t}}{\sqrt{\left(\omega_o^2 - \overline{\omega}^2\right)^2 + \left(2\xi\beta\overline{\omega}\omega_o\right)^2}} e^{-i\theta} - \frac{e^{-i\overline{\omega}t}}{\sqrt{\left(\omega_o^2 - \overline{\omega}^2\right)^2 + \left(2\xi\beta\overline{\omega}\omega_o\right)^2}} e^{+i\theta}\right]$$

$$= \frac{i\ddot{x}_o}{2\omega_o^2}\left[\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}(e^{i(\bar{\omega}t-\theta)} - e^{-i(\bar{\omega}t-\theta)})\right]$$

$$= \frac{i\ddot{x}_o}{2\omega_o^2}\left(\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}\right)2i\sin(\bar{\omega}t-\theta)$$

$$= -\frac{\ddot{x}_o}{\omega_o^2}\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}\sin(\bar{\omega}t-\theta)$$

Thus, the same expression for the steady state response of the SDOF system subjected to the harmonic earthquake acceleration is obtained using the time and frequency domain approach.

Example 3.2

Show that the displacement response of an undamped SDOF system subjected earthquake acceleration, $\ddot{x}_g(t) = \ddot{x}_0 e^{-at}$ is given by

$$x(t) = -\frac{\ddot{x}_0}{a^2+\omega_0^2}\left[\frac{a}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-at}\right]$$

where, ω_0 = natural frequency of the SDOF system; and a = parameter having the same unit as that of ω_0.

Solution: The displacement response of the SDOF system to earthquake acceleration, $\ddot{x}_g(t) = \ddot{x}_0 e^{-at}$ is expressed in time domain analysis as

$$x(t) = -\int_0^t \ddot{x}_g(\tau)h(t-\tau)\,d\tau$$

$$x(t) = -\frac{\ddot{x}_0}{\omega_0}\int_0^t e^{-a\tau}\sin\omega_0(t-\tau)\,d\tau$$

On integrating by parts,

$$x(t) = -\frac{\ddot{x}_0}{\omega_0}\left[-\frac{e^{-a\tau}}{a}\sin\omega_0(t-\tau)\right]_0^t + \frac{\ddot{x}_0}{\omega_0}\int_0^t \frac{\omega_0}{a} e^{-a\tau}\cos\omega_0(t-\tau)\,d\tau$$

$$x(t) = -\frac{\ddot{x}_0}{\omega_0}\left[\frac{1}{a}\sin\omega_0 t\right] + \frac{\ddot{x}_0}{a}\left[-\frac{e^{-a\tau}}{a}\cos\omega_0(t-\tau)\right]_0^t + \frac{\ddot{x}_0\omega_0}{a^2}\int_0^t e^{-a\tau}\sin\omega_0(t-\tau)\,d\tau$$

$$x(t) = -\frac{\ddot{x}_g}{\omega_0}\left[\frac{1}{a}\sin\omega_0 t\right] - \frac{\ddot{x}_0}{a^2}e^{-at} + \frac{\ddot{x}_0}{a^2}\cos\omega_0 t - \frac{\omega_0^2}{a^2}x(t)$$

$$x(t)\left[1+\frac{\omega_0^2}{a^2}\right] = -\frac{\ddot{x}_0}{a^2}\left[\frac{a}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-at}\right]$$

$$x(t) = -\frac{\ddot{x}_0}{a^2+\omega_0^2}\left[\frac{a}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-at}\right]$$

Alternate Solution by Direct Solution of Differential Equation

The differential equation of motion of an SDOF system is expressed as

$$\ddot{x}(t) + \omega_0^2 x(t) = -\ddot{x}_g(t)$$

Let the solution of the equation be

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the solution of homogeneous part of differential equation and $x_p(t)$ is the particular solution.

The homogeneous solution will take the following form

$$x_h(t) = \mathrm{A}\cos\omega_0 t + \mathrm{B}\sin\omega_0 t$$

The particular solution of the given differential equation will be

$$x_p(t) = ce^{-at}$$

$$\dot{x}_p(t) = -a\ c\ e^{-at}$$

$$\ddot{x}_p(t) = a^2 c e^{-at}$$

Substituting the above in the equation, $\ddot{x}(t) + \omega_0^2 x(t) = -\ddot{x} e^{-at}$

$$a^2 c e^{-at} + \omega_0^2 c e^{-at} = -\ddot{x}_0 e^{-at}$$

$$c = -\frac{\ddot{x}_0}{a^2 + \omega_0^2}$$

$$x_p(t) = -\frac{\ddot{x}_0}{a^2 + \omega_0^2} e^{-at}$$

The response of the system will be

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{\ddot{x}_0}{a^2 + \omega_0^2} e^{-at}$$

The initial conditions for system with at rest are i.e. at t = 0 are $x(0) = 0$ and $\dot{x}(0) = 0$.

$x(0) = 0$ ➔ $A - \dfrac{\ddot{x}_0}{a^2 + \omega_0^2} = 0$

$$A = \frac{\ddot{x}_0}{a^2 + \omega_0^2}$$

Similarly,

$\dot{x}(0) = 0$ ➔ $-\omega_0 A \times 0 + \omega_0 B \times 1 + \dfrac{a\ddot{x}_0}{a^2 + \omega_0^2} e^{-a\times 0} = 0$

$$B = -\frac{a}{\omega_0}\frac{\ddot{x}_0}{a^2 + \omega_0^2}$$

Substitute the expression for *A* and *B*, the response of SDOF system simplifies to

$$x(t) = \frac{\ddot{x}_0}{a^2 + \omega_0^2}\cos\omega_0 t + \frac{-a}{\omega_0}\frac{\ddot{x}_0}{a^2 + \omega_0^2}\sin\omega_0 t - \frac{\ddot{x}_0}{a^2 + \omega_0^2} e^{-at}$$

$$= -\frac{\ddot{x}_0}{a^2 + \omega_0^2}\left[\frac{a}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-at}\right]$$

Example 3.3

Using the frequency domain approach and time domain analysis, show that the displacement of an undamped SDOF system subjected to earthquake acceleration, $\ddot{x}_g(t) = c_0\delta(t)$ is given by

$$x(t) = -\frac{c_0}{\omega_0}\sin(\omega_0 t)$$

where ω_0 is the natural frequency of the SDOF system and $\delta(t)$ is the Dirac delta function.

Solution:

A. Frequency Domain Analysis

The displacement of SDOF system to earthquake excitation is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ddot{X}_g(\omega) H(\omega) e^{i\omega t} d\omega$$

where,

$$H(\omega) = \frac{1}{\omega_0^2 - \omega^2} \qquad \text{(For undamped system)}$$

And Fourier transform of $\ddot{x}_g(t)$ is

$$\ddot{X}_g(\omega) = \int_{-\infty}^{\infty} -\ddot{x}_g(t)\, e^{-i\omega t} dt$$

$$\ddot{X}_g(\omega) = -\int_{-\infty}^{\infty} c_0 \delta(t) e^{-i\omega t} dt$$

$$\ddot{X}_g(\omega) = -c_0 e^{-i\omega t}\Big|_{t=0}$$

$$\ddot{X}_g(\omega) = -c_0$$

Therefore, the response of the SDOF system is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-c_0)\frac{1}{\omega_0^2 - \omega^2} e^{i\omega t} d\omega$$

$$= -\frac{c_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2}\, d\omega$$

The integral have two poles at $\omega = \pm\ \omega_0$. Applying Cauchy residual theorem for solving the integral

$$x(t) = -\frac{c_0}{2\pi}\left[2\pi i \lim_{\omega\to\omega_0} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2}(\omega - \omega_0) + 2\pi i \lim_{\omega\to\omega_0} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2}(\omega + \omega_0)\right]$$

$$x(t) = -ic_0\left[-\frac{e^{i\omega_0 t}}{2\omega_0} + \frac{e^{-i\omega_0 t}}{2\omega_0}\right]$$

$$x(t) = \frac{ic_0}{2\omega_0}\left[e^{i\omega_0 t} - e^{-i\omega_0 t}\right]$$

$$x(t) = \frac{ic_0}{2\omega_0}[2i \sin \omega_0 t]$$

$$x(t) = \frac{i^2 c_0}{\omega_0} \sin \omega_0 t$$

$$x(t) = -\frac{c_0}{\omega_0} \sin(\omega_0 t)$$

B. Time Domain Approach

$$x(t) = -\int_0^t h(t-\tau)\ \ddot{x}_g(\tau)\, d\tau$$

where,

$$h(t-\tau) = \frac{\sin \omega_0 (t-\tau)}{\omega_0}$$

$$\ddot{x}_g(\tau) = c_0\ \delta(\tau)$$

Therefore, the response of the SDOF system is given by

$$x(t) = -\int_0^t \frac{\sin \omega_0 (t-\tau)}{\omega_0} c_0 \delta(\tau)\, d\tau$$

$$x(t) = \left[-\frac{\sin \omega_0 (t-\tau)}{\omega_0} c_0\right]\Bigg|_{\tau=0}$$

$$x(t) = -\frac{c_0}{\omega_0} \sin \omega_0 t$$

Example 3.4

Show that the maximum displacement response of a damped SDOF system subjected earthquake acceleration, $\ddot{x}_g(t) = \dot{x}_{g0}\delta(t)$ is

$$x_{\max} = -\frac{\dot{x}_{g0}}{\omega_0}\exp\left[-\frac{\xi}{\sqrt{1-\xi^2}}\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$

where ω_0 and ξ are natural frequency and damping ratio of the SDOF system, respectively; $\dot{x}_{g0}$ is the increment in velocity or the magnitude of acceleration impulse and $\delta(t)$ is the Dirac delta function.

Solution: The general solution of SDOF system will be given by

$$\begin{aligned} x(t) &= -\int_0^t \ddot{x}_g(\tau)\,h(t-\tau)\,d\tau \\ &= -\int_0^t \dot{x}_{g0}\delta(\tau)\,\frac{1}{\omega_d}e^{-\xi\omega_0(t-\tau)}\sin\left(\omega_d(t-\tau)\right)d\tau \\ &= -\frac{\dot{x}_{g0}}{\omega_d}e^{-\xi\omega_0 t}\sin\left(\omega_d t\right) \end{aligned}$$

For the maximum displacement of the system, $\dot{x}(t) = 0$

i.e. $$-\frac{\dot{x}_{g0}}{\omega_0}e^{-\xi\omega_0 t}\left[-\xi\omega_0\sin\omega_d t + \omega_d\cos\omega_d t\right] = 0$$

i.e. $$\tan(\omega_d t) = \frac{\omega_d}{\xi\omega_0} = \frac{\omega_0\sqrt{1-\xi^2}}{\xi\omega_0} = \frac{\sqrt{1-\xi^2}}{\xi}$$

From above equation, the time at which maximum displacement occurs will be

$$t_m = \frac{1}{\omega_d}\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

If $\omega_d t_m = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$, it implies that the $\sin\omega_d t_m = \sqrt{1-\xi^2}$ (refer Figure 3.9)

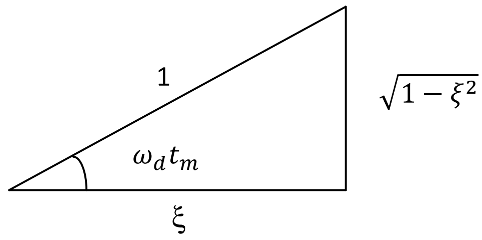

Figure 3.9

The maximum displacement will be given by

$$x_{\max} = -\frac{\dot{x}_{g0}}{\omega_d} e^{-\xi\omega_0 t_m} \sin(\omega_d t_m)$$

$$= -\frac{\dot{x}_{g0}}{\omega_d} \exp\left[-\xi\omega_0 \frac{1}{\omega_d} \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right] \times \sqrt{1-\xi^2}$$

Substituting for $\omega_d = \omega_0\sqrt{1-\xi^2}$ and simplifying

$$x_{\max} = -\frac{\dot{x}_{g0}}{\omega_0} \exp\left[-\frac{\xi}{\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$

Example 3.5

Find the response of SDOF system having time period as 1 sec and damping ratio as 0.02 subjected to the El-Centro, 1940 earthquake motion (refer Appendix-I for the digitized acceleration values). Plot the displacement response of the SDOF system using (a) Exact method of time domain analysis, (b) Newmark's Beta method, and (c) Runge-Kutta method.

Solution:

Based on the computer program written in the FORTAN language, the response of the SDOF system with time period as 1 sec and damping ratio as 0.02 subjected to the El-Centro, 1940 earthquake motion were obtained and is plotted in Figure 3.10. The calculated maximum displacement of the system is found to be 0.15163m, 0.15166m and 0.15158m for exact method of time domain analysis, Newmark's Beta method and Runge-Kutta method, respectively. As expected all the methods predict the same response of the system. Further, time interval taken for numerical integration of equation of motion of the system is 0.002 sec for Newmark's Beta and Runge-Kutta methods.

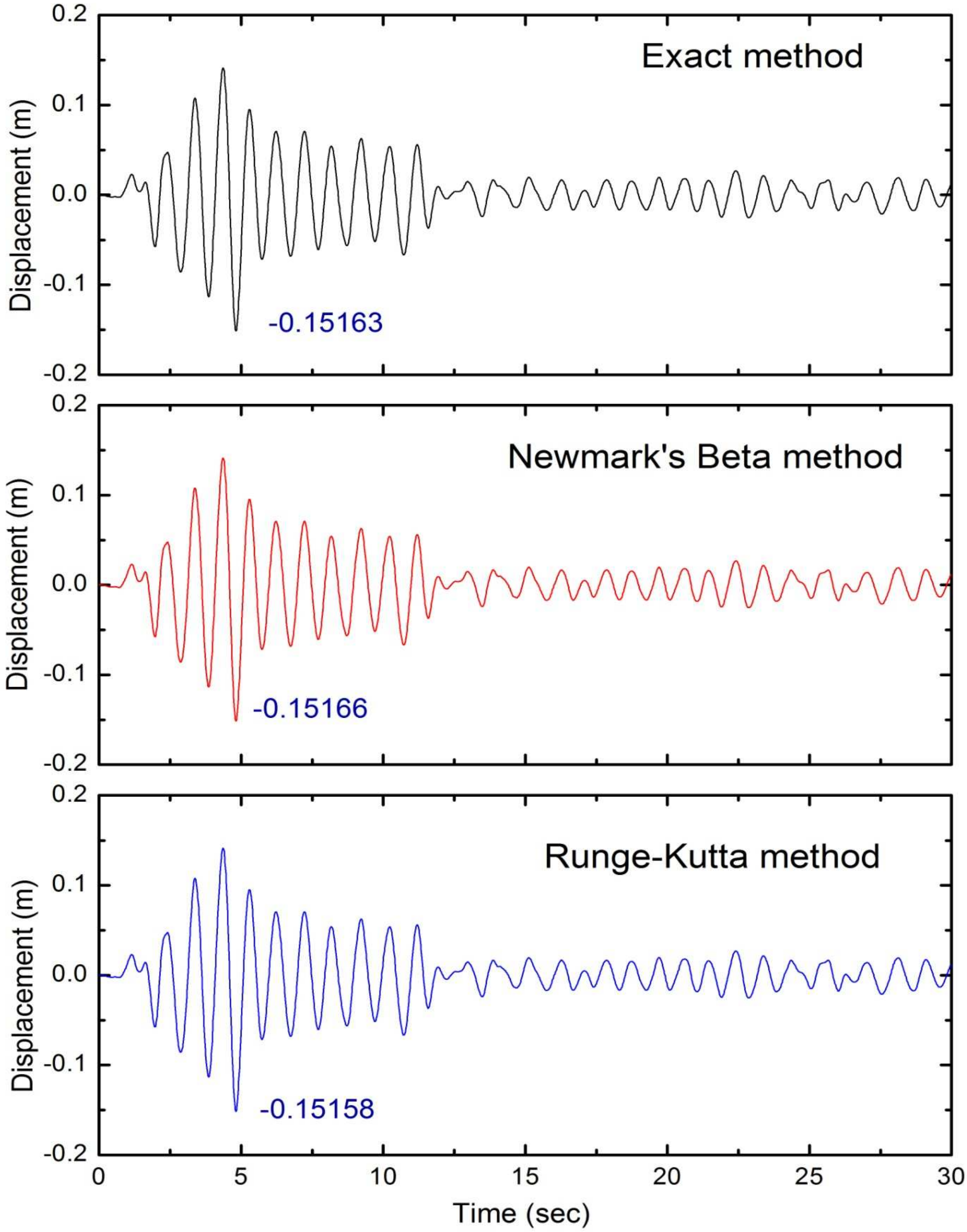

Figure 3.10 Response of SDOF system of Example 3.5 by various methods.

3.5 Response Analysis of MDOF System

Multi degree of freedom (MDOF) systems are usually analyzed using modal superposition analysis. A typical MDOF system with n degrees of freedom is shown in Figure (3.11). This system when subjected to ground motion undergoes deformations in number of possible ways. These deformed shapes are known as modes of vibration or mode shapes. Each shape is vibrating with a particular natural frequency. Total unique modes for each MDOF system are equal to the possible degrees of freedom of system. The equations of motion for MDOF system is given by

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m]\{r\}\ddot{x}_g(t) \tag{3.64}$$

where, $[m]$ = Mass matrix ($n \times n$); $[k]$ = Stiffness matix ($n \times n$); $[c]$ = Damping matrix ($n \times n$); $\{r\}$ = Influence coefficient vector (n×1); $\{x(t)\}$ = relative displacement vector; $\{\dot{x}(t)\}$ = relative velocity vector, $\{\ddot{x}(t)\}$ = relative acceleration vector, and $\ddot{x}_g(t)$ = earthquake ground acceleration.

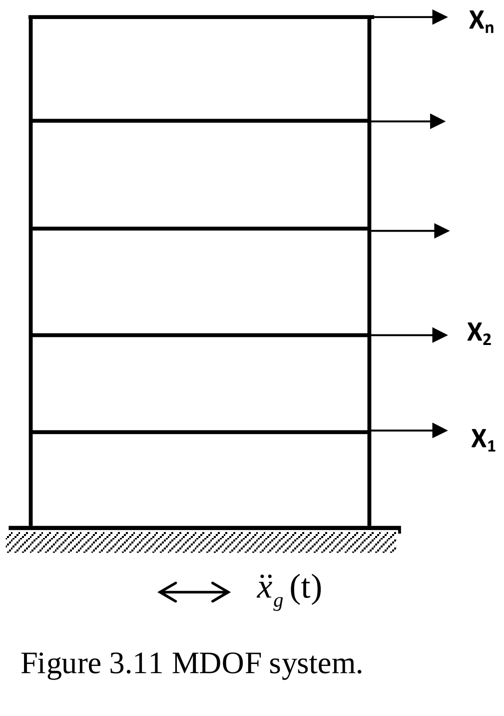

Figure 3.11 MDOF system.

The undamped eigen values and eigen vectors of the MDOF system are found form the characteristic equation

$$\left\{[k] - \omega_i^2 [m]\right\}\phi_i = 0 \qquad i = 1,2,3,\dots,n \tag{3.65}$$

$$\det\left|\left\{[k] - \omega_i^2 [m]\right\}\right| = 0 \tag{3.66}$$

where,

ω_i^2 = eigen values of the ith mode

ϕ_i = eigen vector or mode shape of the ith mode

ω_i = natural frequency in the ith mode.

Let the displacement response of the MDOF system is expressed as

$$\{x(t)\} = [\phi]\{y(t)\} \tag{3.67}$$

where $\{y(t)\}$ represents the modal displacement vector, and $[\phi]$ is the mode shape matrix given by

$$[\phi] = [\ \phi_1, \phi_2\ ,\dots\dots\dots\ \phi_n\ ] \tag{3.68}$$

Substituting $\{x\} = [\phi]\{y\}$ in equation (3.64) and pre-multiply by $[\phi]^T$

$$[\phi]^T[m][\phi]\{\ddot{y}(t)\} + [\phi]^T[c][\phi]\{\dot{y}(t)\} + [\phi]^T[k][\phi]\{y(t)\} = -[\phi]^T[m]\{r\}\ \ddot{x}_g(t) \tag{3.69}$$

The above equation reduces to

$$[M_m]\{\ddot{y}(t)\} + [C_d]\{\dot{y}(t)\} + [K_d]\{y(t)\} = -[\phi]^T[m]\{r\}\ \ddot{x}_g(t) \tag{3.70}$$

where,

$[\phi]^T[m][\phi] = [M_m]$ = generalized mass matrix

$[\phi]^T[c][\phi] = [C_d]$ = generalized damping matrix

$[\phi]^T[k][\phi] = [K_d]$ = generalized stiffness matrix

By virtue of the properties of the [ϕ], the matrices [M_{m}] and [K_{d}] are diagonal matrices. However, for the classically damped system (i.e. if the [C_{d}] is also a diagonal matrix), the equation (3.70) reduces to the following equation

$$\ddot{y}_i(t) + 2\xi_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = -\Gamma_i\ \ddot{x}_g(t) \qquad (i = 1, 2, 3, \ldots, n) \tag{3.71}$$

where,

$y_i(t)$ = modal displacement response in the i$^{\text{th}}$ mode,

ξ_i = modal damping ratio in the i$^{\text{th}}$ mode, and

Γ_i = modal participation factor for i$^{\text{th}}$ mode expressed by

$$\Gamma_i = \frac{\{\phi_i\}^T[m]\{r\}}{\{\phi_i\}^T[m]\{\phi_i\}} \tag{3.72}$$

The equation (3.71) represents *n* second order differential equations (i.e. similar to that of a SDOF system) and the solution of which will provide the modal displacement response in the i$^{\text{th}}$ mode, $y_i(t)$ for *i*=1 to *n*. The displacement response of the MDOF system can be obtained by equation (3.67) using the {$y(t)$}. The other response quantities of the structure can be obtained from the displacement response of the system.

3.6 Numerical Examples on MDOF System

Example 3.6

A two-story building is modeled as 2-DOF system and rigid floors as shown in the Figure 3.12. Determine the top floor maximum displacement and base shear due to El-Centro, 1940 earthquake ground motion. Take the inter-story stiffness, k =197.392 × 10^3 N/m, the floor mass, m = 2500 kg and damping ratio as 2%.

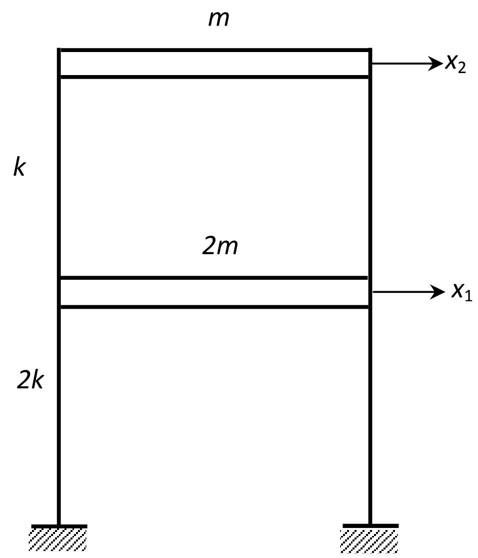

Figure 3.12

Solution:

Mass of each floor, m = 2500 kg and stiffness, k = 197.392 kN/m

$$\text{Stiffness matrix} = [k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$$

$$\text{Mass matrix} = [m] = \begin{bmatrix} 5000 & 0 \\ 0 & 2500 \end{bmatrix}$$

Using the equations (3.65) and (3.66), the frequencies and mode-shapes of the structures are

$\omega_1 = 6.283$ rad/sec and $\omega_2 = 12.566$ rad/sec

$$\{\phi_1\} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad \text{and} \quad \{\phi_2\} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The modal column matrix is given by

$$[\phi] = [\phi_1 \ \phi_2] = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$

The modal participation factors are given by

$$\Gamma_i = \frac{\{\phi_i\}^T [m]\{r\}}{\{\phi_i\}^T [m]\{\phi_i\}}$$

$$\Gamma_1 = \frac{\{\phi_1\}^T [m]\{r\}}{\{\phi_1\}^T [m]\{\phi_1\}} = \frac{[0.5 \quad 1]\begin{bmatrix} 5000 & 0 \\ 0 & 2500 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{[0.5 \quad 1]\begin{bmatrix} 5000 & 0 \\ 0 & 2500 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}} = 1.333$$

Similarly,

$$\Gamma_2 = \frac{\{\phi_2\}^T [m]\{r\}}{\{\phi_2\}^T [m]\{\phi_2\}} = \frac{[-1 \quad 1]\begin{bmatrix} 5000 & 0 \\ 0 & 2500 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{[-1 \quad 1]\begin{bmatrix} 5000 & 0 \\ 0 & 2500 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = -0.333$$

The response in the each mode of vibration is computed by solving the Equation (3.71) for the system. The displacement and base shear response is shown in the Figures 3.13 and 3.14, respectively. The maximum top floor displacement and base shear are found to be 0.202 m and 40.72 kN, respectively.

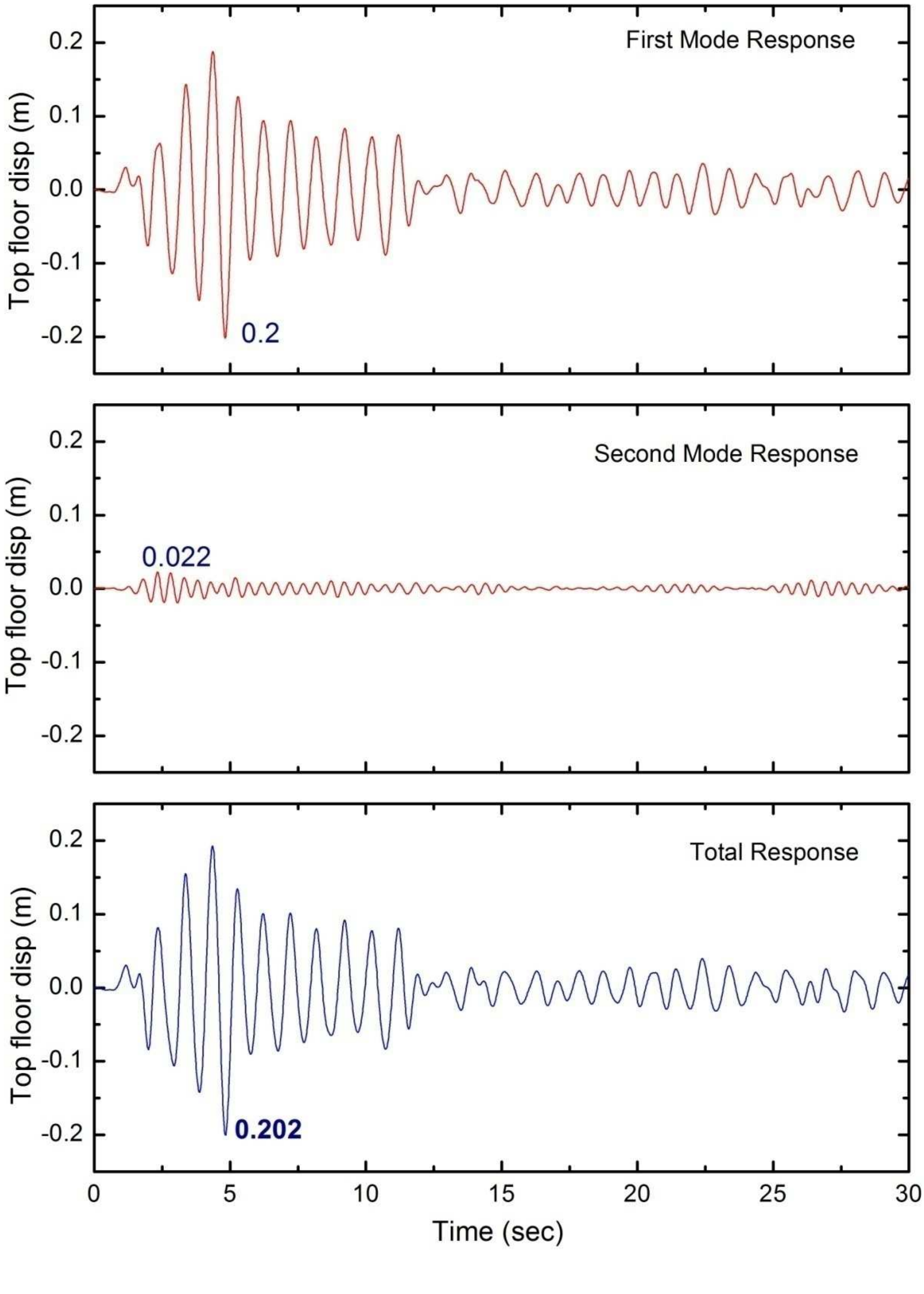

Figure 3.13 Top floor displacement response of two DOF system of Example 3.6.

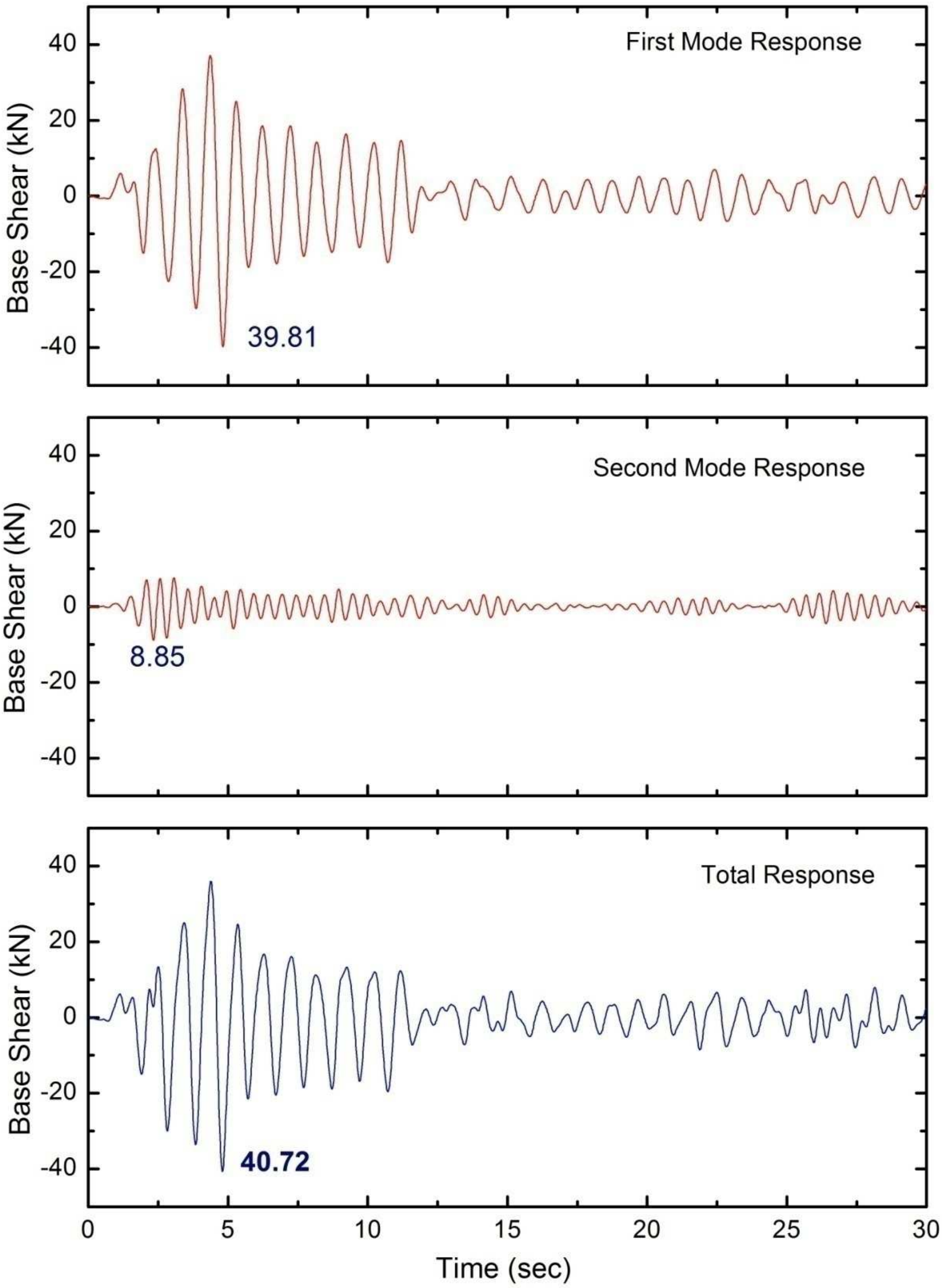

Figure 3.14 Base shear response of two DOF system of Example 3.6.

Example 3.7

An industrial structure is modeled as 2-DOF system as shown in the Figure 3.15. Determine the horizontal and vertical displacement of the free end of the structure due to El-Centro, 1940 earthquake ground motion. Take EI =80 × 10^3 N.m^2, L= 2m, m_1= 100kg and m_2= 200kg. The damping shall be considered as 2 percent.

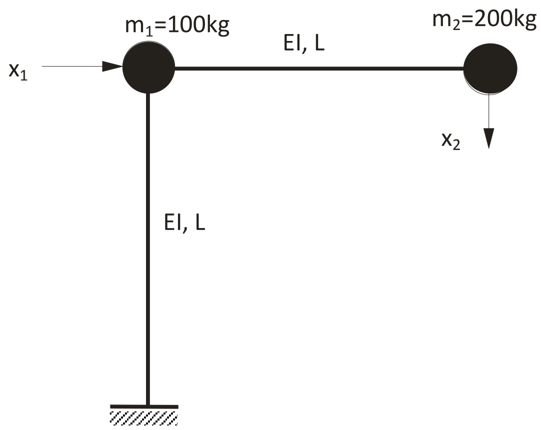

Figure 3.15

Solution: Given,

Mass, m_1= 100kg, m_2= 200kg, Length, L= 2 m and flexural rigidity, EI= 80 × 10^3 Nm^2

Stiffness matrix $=[k]=\dfrac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$;

Mass matrix = $[m]=\begin{bmatrix} 300 & 0 \\ 0 & 200 \end{bmatrix}$

Using equation (3.65), eigen values and eigen vectors can be obtained as

ω_1 = 5.4925 rad/sec; ω_2 = 16.856 rad/sec

$\{\phi_1\}=\begin{Bmatrix} 2.7 \\ 6.25 \end{Bmatrix}$; $\{\phi_2\}=\begin{Bmatrix} 5.103 \\ -3.307 \end{Bmatrix}$

Modal participation can be obtained by

$$\Gamma_i = \frac{\{\phi_i\}^T [m]\{r\}}{\{\phi_i\}^T [m]\{\phi_i\}}$$

$\Gamma_1 = 0.081$ and $\Gamma_2 = 0.153$

The displacement response in the each mode of vibration is computed by solving the Equation (3.71) for the system. The horizontal and vertical displacement of the free end of the structure is shown in the Figures 3.16 and 3.17, respectively. The maximum horizontal and vertical displacement of the free end of the structure is found to be 0.039 m and 0.0699 m, respectively.

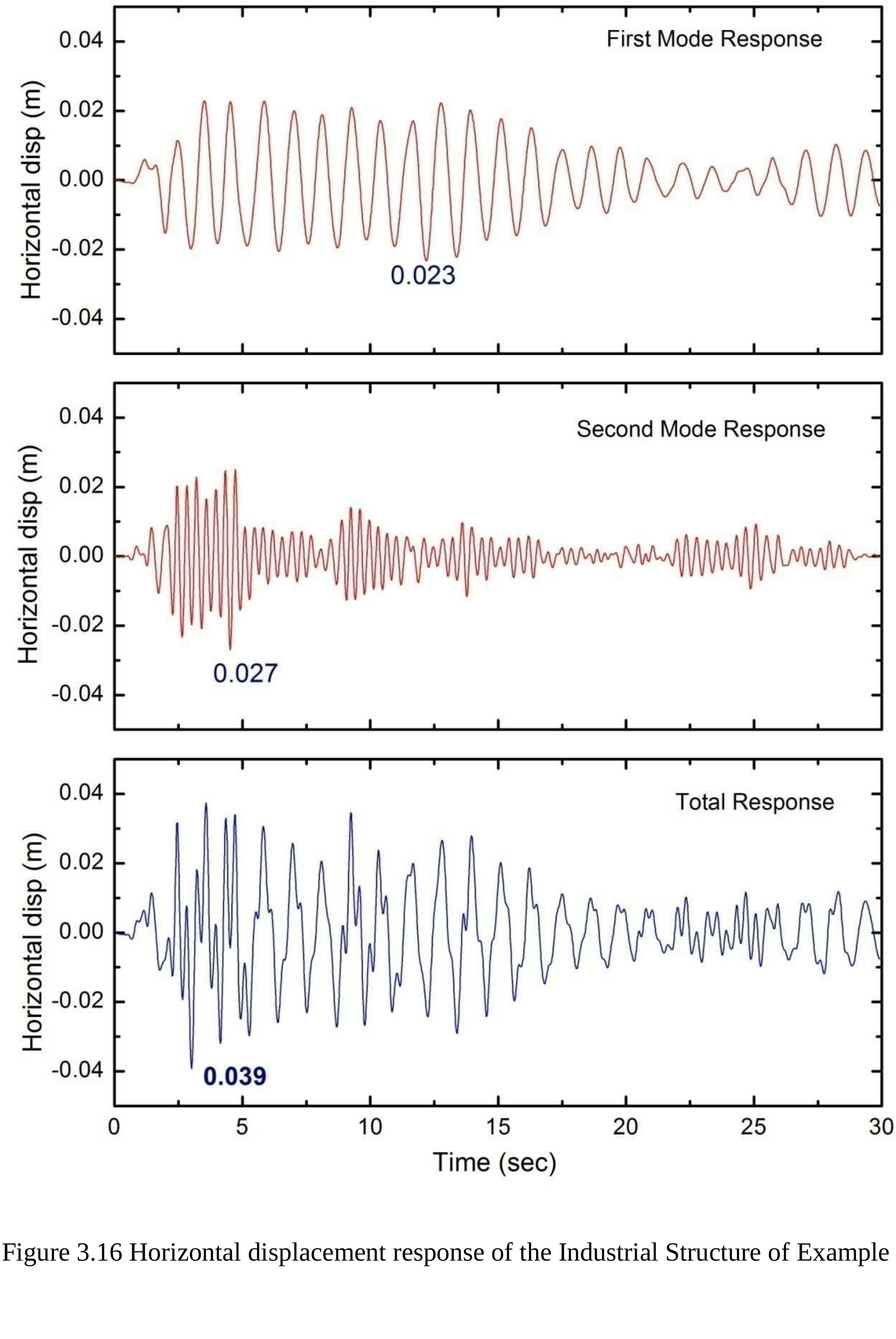

Figure 3.16 Horizontal displacement response of the Industrial Structure of Example 3.7.

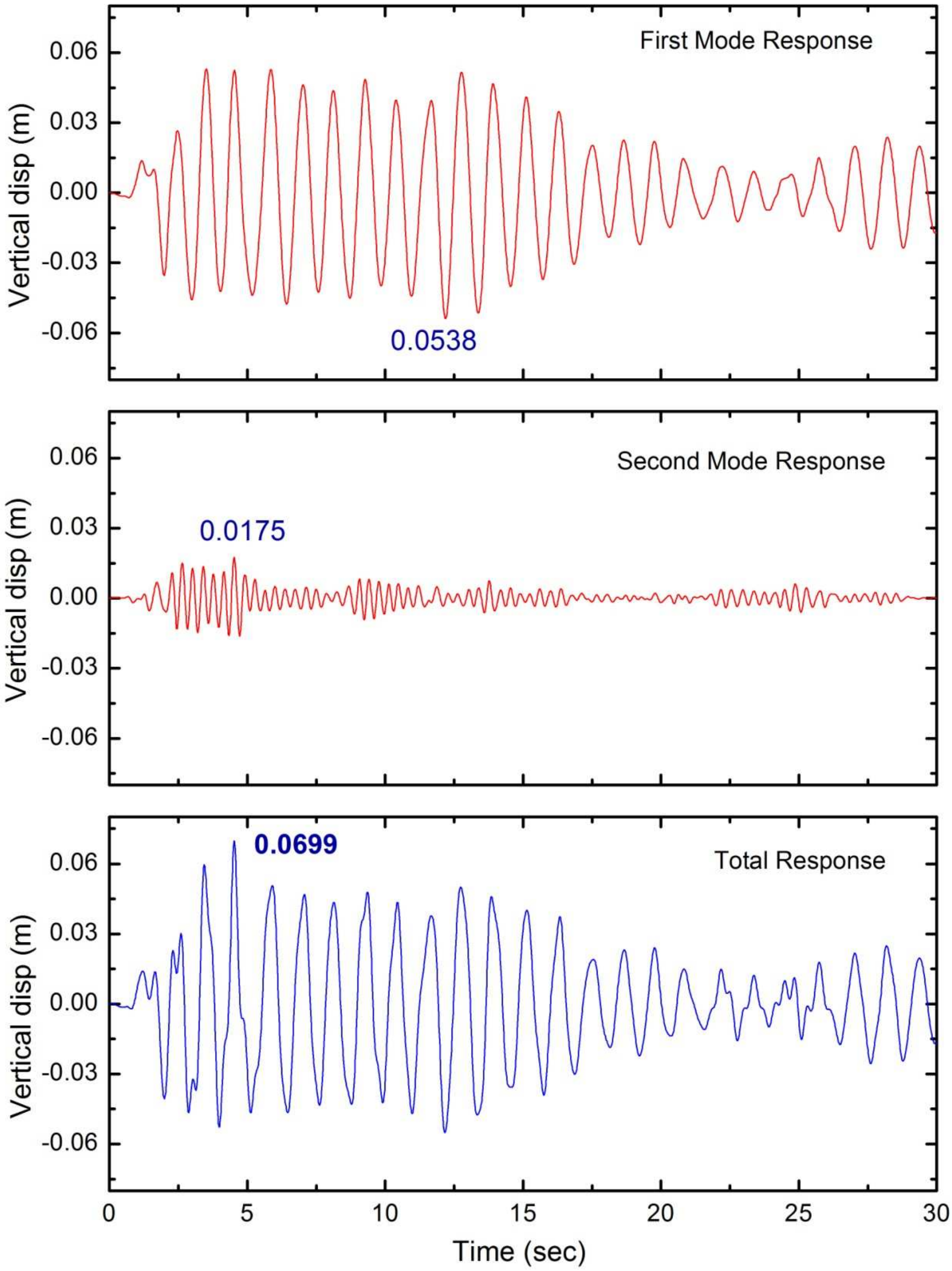

Figure 3.17 Vertical displacement response of the Industrial Structure of Example 3.7.

3.8 Tutorial Problems

Q1. Develop general computer programs (preferably using Matlab or Scilab) to obtain the response of a SDOF system under earthquake excitation using (a) Newmark's Beta method, (b) Runge-Kutta method, and (c) Exact method of time domain analysis. Compare the results from above three methods by plotting the response of a SDOF system having time period as 0.5 sec and damping ratio as 0.05 under the El-Centro, 1940 motion.

Q2. Derive the expressions for the elements of matrices [A] and [B] (i.e. equations (3.28) and (3.29)) used in exact method for evaluation of the response of a SDOF system under earthquake excitation.

Q3. Derive the expression for displacement response of an undamped SDOF system subjected to earthquake ground motion of $\ddot{x}_g(t) = \ddot{x}_o(e^{-at} - e^{-bt})$. Take, ω_0 = natural frequency of the SDOF system; and a = parameter having the same unit as that of ω_0.

Q4. A rigid-jointed plane frame is fixed at A and roller support at C as shown in Figure 3.18. The members AB and BC are rigidly connected at B making a right angle and are supporting a mass of 200 kg. Neglect the mass of frame, determine the maximum horizontal displacement and base shear due to El-Centro, 1940 earthquake. Take the flexural rigidity, EI = 294772.2 Nm^2 and length, L = 4m for both members. Consider the damping as 2%.

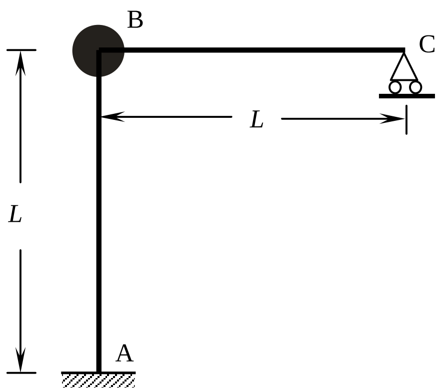

Figure 3.18

Q5. A 2-degrees-of-freedom system (Figure 3.19) is subjected to horizontal earthquake excitation of El-Centro, 1940 earthquake. Take the flexural rigidity, EI = 10^6 Nm^2 and length, L = 2m. The each lumped mass is 100 kg. Determine the maximum displacement of the two masses. Take 2% damping in each mode of vibration.

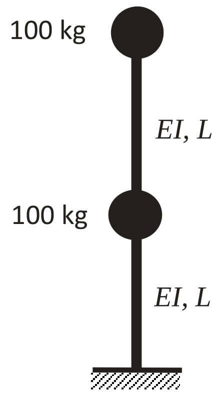

Figure 3.19

Q6. A three-story building is modeled as 3-DOF system and rigid floors as shown in Figure 3.20. Determine the maximum top floor maximum displacement and base shear due to El-Centro, 1940 earthquake ground motion. Take the inter-story lateral stiffness of floors i.e. k_1 = k_2= k_3=16357.5 kN/m, the floor mass m_1= m_2=10000 kg and m_3=5000 kg and damping ratio as 2%.

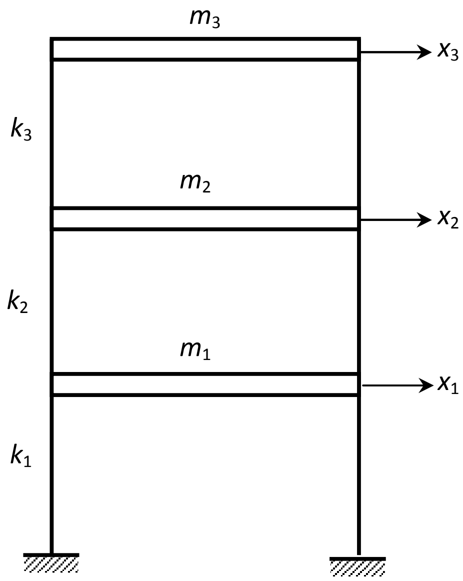

Figure 3.20

3.8 Answers to Tutorial Problems

Q1. Maximum Displacement = 0.057 m

Q3.

$$x(t) = -\frac{\ddot{x}_0}{a^2+\omega_0^2}\left[\frac{a}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-at}\right] + \frac{\ddot{x}_0}{b^2+\omega_0^2}\left[\frac{b}{\omega_0}\sin\omega_0 t - \cos\omega_0 t + e^{-bt}\right]$$

Q4. Horizontal displacement = 0.0684 m

Base shear = 2160.2 N

Q5. Displacement of lower mass = 7.76×10^{-3} m

Displacement of top mass = 24.02×10^{-3} m

Q6. Top floor displacement = 0.0234 m

Base shear = 196.4×10^3 N